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Study of Nucleon Induced Reactions at Intermediate Energies
I. Medium Effects

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CONTENTS

Abstract

This work is dedicated to study the in-medium effects in proton-nucleus collisions at intermediate energy ~١ GeV within the framework of the microscopic transport ultrarelativistic quantum molecular dynamic (UrQMD) model.

At intermediate energy the neutron spectra in pA collisions have two peaks at very forward angles, referred to as quasielastic and quasi-inelastic peaks.

The quasielastic peak, centered near the beam energy, is attributed to a single (p,n) elastic scattering. The quasi-inelastic peak, located at the beam energy minus ≈ ٣٠٠ MeV, is associated with the excitation of Δ- resonance in inelastic (p,n) scattering. Until now a comprehensive theoretical description of the location and intensity of these two peaks is not yet achieved. Therefore, it seems to us that it is worthwhile to make a further study by inclusion of medium effects in the dynamical content of the microscopic transport UrQMD model. This includes, mean field effect and the in-medium angular distributions of $NN - NN$, $NN - N\Delta$, and $N\Delta - NN$ processes.

It is shown that the introduction of the in-medium effects in the UrQMD calculations improves the intensity and location of these two peaks.

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Chapter ١

Introduction

One of the main interests of heavy-ion physics and astrophysics is the property of nuclear matter under extreme conditions. Its high density behaviour is important for the scenario of supernova explosions, the evolution of neutron stars, the reaction process of high energy heavy-ion collisions, quark-gluon plasma, and so on. The high energy nuclear collision is the only experimentally accessible way to create such a kind of hot and dense nuclear matter in the laboratory and provides us with a unique opportunity to study it.

Proton-nucleus (pA) and nucleus-nucleus (AA) collisions in the GeV/nucleon range are generally analyzed in terms of transport models. There are mainly three categories of models. The models in the first category assume Glauber geometry for the treatment of AA collisions [١-٧]. In these models, main quantum features during the multiple scattering are preserved and efficiently fast calculations are

possible. However, these approaches are mainly designed for the extremely high energy collisions ($\sqrt{s}=$١٠ AGeV), where $\sqrt{s}$ is the center-of-mass (C.M.) energy.

The models in the second category (parton cascade models) [٨،٩], have been recently developed to implement the interaction among partons to study space-time evolution of partons produced in high energy nuclear collisions. These models have been designed originally to describe ultra-relativistic heavy-ion collisions at collider energies (≥٢٠٠ GeV) [١٠].

The third category of models is transport model which is often referred to as 'hadronic cascade' [١١-١٨]. For example, the relativistic quantum molecular dynamics (RQMD) [١١،١٢], quark gluon string (QGS) [١٣] and ultra-relativistic quantum molecular dynamics (UrQMD) [١٦،١٧] models. They have been successfully used to describe many aspects of high energy heavy-ion collisions in a wide range of incident energies.

As is well known in high energy nuclear collisions, a drastic density is exhibited, that is to say, as the energy increases the density of matter produced in high energy nuclear collisions

becomes higher. Consequently, the properties of the particles like effective-mass, in-medium cross sections and in-medium angular distributions might change significantly. Thus the role of medium effects on the two-body scattering becomes important and should be taken into account.

The aim of this work is to study 'in-medium' and 'mean field' effects in the framework of the UrQMD model. The former only includes modifications to the NN-*elastic* and $NN - N\Delta$ angular distributions as well as changes to the Δ- mass distribution. The latter incorporates the quantum molecular dynmics (QMD [١٩]) interaction potential between nucleons.

The influence of these two effects on the quasielastic and inelastic peaks of neutrons in pA is studied. The quasielastic peak, centred near the incident beam energy, has been interpreted as quasielastic charge exchange nucleon-nucleon (NN) collisions inside the target nucleus. The quasi-inelastic peak, located at the beam energy minus $\approx$ ٣٠٠ MeV, is associated with the excitation of Δ- resonance in inelastic NN-collisions. In contrast with the quasielastic peak, which can be attributed to a single (p,n) elastic

scattering, in the quasi-inelastic region, the single (p,n) inelastic scattering contribution is superimposed to a background of multiple scattering contribution.

It should be noted that the current microscopic models (see, e.g. [٢٠-٢٧]) are having difficulty of reproducing the location and intensity of the quasielastic and inelastic peaks at intermediate energies.

The present work is organized as follows, In chapter ٢, we will give a brief description of the basic principles of the UrQMD model mainly at intermediate energies. In chapter ٣ predictions of the model using free space and medium modified NN-*elastic* and $NN - N\Delta$ angular distributions are compared with one another and with the recent measurements of double differential neutron production cross sections as a function of neutron kinetic energy (E_n) at ٠° and ١٠° for $p+$ ٢٧*Al*, ٥٦*Fe*, and ٩١*Zr* at ١,٢ GeV. Finally, we summarize and conclude this work in chapter ٤.

Chapter ٢

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Description of the model used

A microscopic dynamic description of pA and AA collisions is usually based on transport theory. Here a sequence of propagations of particles is simulated numerically. This includes baryons (resonances) and mesons with or without interaction (potential) and subsequent scattering processes or decays. The main ingredients in this description of pA and AA collisions are the cross sections, the two-body potential and decay width.

It is obvious that - at least in principal - these models should be obtained consistently from underlying theory. Since the particles propagate in a dense and not-equilibrated medium of excited fragments, the properties of the particles might change significantly.

Consequently properties like effective-mass, in-medium cross sections and in-medium angular distributions should be taken into account.

Unfortunately, this is a very complicated task, so that most of the current models adopt free-cross sections and free-interactions.

Furthermore, transport models, such as Boltzmann Uehling-Uhlenbeck (BUU or VUU) type [٢٨] for the nuclear phase-space distribution: solving either a non-relativistic transport equation or a relativistic transport equation (RBUU) [٢٩], describe the system by the one-body phase distribution function, which does not contain any information about correlations of particles. In the QMD models [١٩], this shortcoming is avoided, since the particles can interact by individual two-body forces.

The generalization of such two-body forces to the relativistic region is, however, not a simple task. In principle, the interaction must be mediated by fields, which are propagated according to wave equations or one must make use of the so-called constrained Hamiltonian dynamics.

In the present work, we discuss in detail one specific microscopic transport model, the UrQMD* model [١٧]. In contrast to

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* A web page which contains the structure of the UrQMD code can be found in http://www.th.physik.uni-frankfurt.de/~urqmd/.

other microscopic models [١٩،٢٨-٣٢], it incorporates several advantages:

i. It includes all baryonic and mesonic resonances as tabulated by the particle data group [٣٣].

ii. A two-body (mean field) dynamics.

iii. Medium-modified angular distribution for NN- elastic scattering.

In this chapter, we describe the main ingredients of the UrQMD model such as collision criteria (Sec. ٢,٢), cross sections (Sec. ٢,٣), and effective two-body interactions (Sec. ٢,٥). A detailed description of the UrQMD ingredients can be found in Ref. [١٧].

Section ٢,٤ focuses on the free and medium modified differential cross sections which will be applied in our numerical calculations.

٢,١ ***Initial conditions of nuclear collision***

In UrQMD the two colliding nuclei are assumed to move along the z-direction until the distance between their centers is $\vec{b}$, where $\vec{b}$ is the impact parameter. The initial positions (x_i, y_i, z_i) and momenta (p_{xi}, p_{yi}, p_{zi}) of the projectile (target) nucleons are randomly distributed within a sphere with spherical polar coordinates $(R(p_F^{\max}), \theta, \varphi)$

$$\begin{cases} x_i = R\sin\theta\cos\varphi \\ y_i = R\sin\theta\sin\varphi \\ z_i = R\cos\theta, \end{cases} \qquad \begin{cases} p_{xi} = p_F^{\max}\sin\theta\cos\varphi \\ p_{yi} = p_F^{\max}\sin\theta\sin\varphi \\ p_{zi} = p_F^{\max}\cos\theta, \end{cases} \qquad (٢,١)$$

where $i = ١,٢,٣,..., A$ and R is the nuclear radius given by [١٧],

$$R = ١.١٢٤\left[\frac{١}{٢}\left(A + \left(A^{١/٣} - ١\right)^{٣}\right)\right]^{١/٣} \text{fm}, \qquad (٢,٢)$$

where A being the mass number of the nucleus, while $p_F^{\max}$, the maximum Fermi momentum is defined by

$$p_F^{\max} = \hbar c\left(٣\pi^{٢}\rho\right)^{١/٣}, \qquad (٢,٣)$$

where ρ is the nucleon density.

The positions and linear momenta of the two colliding nuclei in their (C.M.) reference frame are determined by the initial positions and

momenta of Eq. (٢,١) and the projectile laboratory (kinetic) energy T:

$$\begin{cases} x_i' = x_i \\ y_i' = y_i \\ z_i' = (z_i + R)/\gamma, \end{cases} \qquad \begin{cases} p_{xi}' = p_{xi} \\ p_{yi}' = p_{yi} \\ p_{zi}' = \gamma(p_{zi} \mp \beta E), \end{cases} \qquad (٢,٤)$$

where $i = ١,٢,٣,..., A.$

The minus and plus signs indicate the direction of the target and projectile along z-axis, respectively.

where: $$\gamma = ١/\sqrt{(١ - \beta^{٢})},$$

$$\beta = p/E\,,$$

$$p = \sqrt{T^{٢} + ٢mT}\,,$$

$$E = T + m\,, \qquad (٢,٥)$$

The indices* E, β, m, and γ stand for the energy, velocity of nucleon, nucleon mass and Lorentz factor, respectively.

In order to simulate Pauli principle in space, a minimum distance of ١,٦ fm is imposed between nucleons. The phase space density at the location of each nucleon is evaluated: if the phase space density is

too high, then the location of that nucleon is rejected and a new location is randomly chosen. This procedure reduces fluctuations in the mean density of the nucleus.

٢,٢ ***The collision criterion***

Nuclear collisions are assumed to be described by the sum of independent binary hadron-hadron (*hh*) collisions. Each *hh* collision is assumed to take place at the distance of closest approach, that is, two particles collide if their distance d_{trans} fulfills the relation:

$$d_{trans} \leq \sqrt{\frac{\sigma_{tot}}{\pi}}, \qquad \sigma_{tot} = \sigma\left(\sqrt{s}, type\right).$$

(٢,٦)

The total cross section σ_{tot} depends on the C.M. energy $\left(\sqrt{s}\right)$, the species, and quantum number of the particles. The distance d_{trans} can be defined as the covariant relative distance between the two particles as

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* We use throughout the units of MeV, fm, and fm/c for energy, length and time, respectively. Let us remind that $\hbar$ = ١٩٧,٣ MeV fm and c=١.

$$d_{trans} = \sqrt{\left(\vec{r}_١ - \vec{r}_٢\right)^٢ - \frac{\left(\vec{r}_١ - \vec{r}_٢\right).\left(\vec{p}_١ - \vec{p}_٢\right)}{\left(\vec{p}_١ - \vec{p}_٢\right)^٢}},$$

(٢,٧)

with $\vec{r}_i$ being the location and $\vec{p}_i$ the momentum in the rest frame of the colliding particles.

٢,٣ Cross sections

The UrQMD uses a table-look-up for the total and elastic proton-proton and proton-neutron cross sections. The details of other *hh*-cross sections implemented in the UrQMD model at ١,٢ GeV incident energy ($\sqrt{s} =$ ٢,٤١ GeV) are given by

١. $B_i + B_j \to B_i + B_j$;

٢. $N + N \to N + \Delta_{١٢٣٢}$;

٣. $N + \Delta_{١٢٣٢} \to N + N$;

٤. $N + N \to N + \Delta^*$;

٥. $N + N \to \Delta_{١٢٣٢} + N^*$;

٦. $N + N \to N + N^*$;

٧. $N + N^* \to N + N$;

٨. $N + \pi \to \Delta_{١٢٣٢}$;

٩. $N + \pi \to N^{*}$;

١٠. $\Delta_{١٢٣٢} + \pi \to N^{*}$;

(٢,٨)

where *B* denotes a baryon, and *N*, more specifically, a nucleon. The $\Delta_{١٢٣٢}$ is explicitly listed, whereas higher excitations of the Δ- resonance have been denoted as Δ^{*}.

For the production of baryonic resonances [channels ٢, ٤, ٥,

١١

and ٦ in Eq. (٢,٨)] the cross sections are parameterized according to the general form,

$$\sigma_{١,٢\to ٣,٤}\left(\sqrt{s}\right) \propto \left(٢S_{٣}+١\right)\left(٢S_{٤}+١\right)\frac{p_{٣,٤}}{p_{١,٢}}\frac{١}{s}\left|M(m_{٣},m_{٤})\right|^{٢}, \qquad (٢,٩)$$

where S_i, i=٣،٤ express the spin of the particles in the final state and $p_{i,j}$ corresponds to the C.M. momentum of the particles (i) and (j). Specific assumptions are made with regard to the form of the matrix element $\left|M(m_{٣},m_{٤})\right|^{٢}$ for each resonance production channel.

For channel ٢ in Eq. (٢,٨),

$$\left|M(m_{٣},m_{٤})\right|^{٢} = A\frac{m_{\Delta}^{٢}\Gamma_{\Delta}^{٢}}{((\sqrt{s})^{٢}-m_{\Delta}^{٢})^{٢}+m_{\Delta}^{٢}\Gamma_{\Delta}^{٢}},$$

(٢,١٠)

is used with m_{Δ}=١٢٣٢ MeV, Γ_{Δ}=١١٥ MeV, and A=٤٠٠٠٠ [١٧].

As for channels ٤, ٥, and ٦ in Eq. (٢,٨),

$$\left|M(m_{٣},m_{٤})\right|^{٢} = A\frac{١}{(m_{٤}-m_{٣})^{٢}(m_{٤}+m_{٣})^{٢}},$$

(٢,١١)

is taken with A=٦,٣ for channel ٦, A=١٢ for channel ٤, and A=٣,٥ for channel ٥ [١٧]. The free parameters in Eqs. (٢,١٠) and (٢,١١) are tuned to experimental measurements.

The cross sections for channels ٣ and ٧ in Eq. (٢,٨) are determined by the law of detailed balance from the cross sections of channels ٢ and ٦, respectively,

$$\sigma_{٣,٤\rightarrow ١,٢} \propto \frac{\left\langle p_{١,٢}^{٢}\right\rangle}{\left\langle p_{٣,٤}^{٢}\right\rangle}\frac{(٢S_{١}+١)(٢S_{٢}+١)}{(٢S_{٣}+١)(٢S_{٤}+١)}\sigma_{١,٢\rightarrow ٣,٤}\,.$$

(٢,١٢)

The integration over the mass distributions of the resonances in Eq. (٢,١٢) has been denoted by the brackets $\langle\ \rangle$, e.g.,

$$\left\langle p_{٣,٤}^{٢}\right\rangle = \iint p_{٣,٤}^{٢}\left(\sqrt{s},m_{٣},m_{٤}\right)A(m_{٣})A(m_{٤})dm_{٣}dm_{٤}\,. \qquad (٢,١٣)$$

The mass distribution $A(m)$ in Eq. (٢,١٣) is given by the Breit-Wigner distribution with a mass dependent width,

$$A(m) = \frac{١}{N}\frac{\Gamma(m)}{(m_R - m)^{٢} + [\Gamma(m)]^{٢}/٤} \ , \qquad (٢,١٤)$$

where N denotes the normalization constant and $\Gamma(m)$ is the mass-dependent width.

In the case of π-absorption on baryons (channels ٨, ٩, and ١٠) the total meson-baryon cross section is given by

$$\sigma(MB \to R) = \sum_{R \to \Delta, N^*} |C(MB,R)|^{٢} \frac{(٢S_R + ١)}{(٢S_M + ١)(٢S_B + ١)} \frac{\pi}{p_{C.M}^{٢}}$$

$$\times \frac{\Gamma_{R \to MB}\Gamma_{tot}}{(\sqrt{s} - m_R)^{٢} + \Gamma_{tot}^{٢}/٤}, \qquad (٢,١٥)$$

Where $C(MB,R)$ are the Clebsch-Gordon coefficients. S_R, S_B, and S_M denote the spin of resonance, the decaying baryon and meson, respectively. The full width Γ_{tot} is a sum of all partial decay width Γ_{R-MB} for resonance R into mesons M and baryons B, which depends on the momentum of the decaying particle,

$$\Gamma_{R \to MB} = \Gamma^{٠}_{R \to MB} \frac{m_R}{m} \left(\frac{p_{C.M}(m)}{p_{C.M}(m_R)} \right)^{٢l+١} \frac{١.٢}{١ + ٠.٢\left(\frac{p_{C.M}(m)}{p_{C.M}(m_R)} \right)^{٢l}}, \qquad (٢,١٦)$$

$\Gamma^{'}_{R-MB}$ is the partial decay width of the resonance into the channel M and B. l and $p_{C.M}(m)$ are the relative angular momentum and the relative momentum in their rest frames, respectively.

The decay of the resonances proceeds according to the branching ratios compiled by the particle data group [٣٣]. The resonance decay products have isotropical distributions in the rest frame of the resonance.

The inelastic hh collisions produce resonances at low and intermediate energies, while at high energies ($\sqrt{s}=$ ٥ GeV for baryon-baryon and ٣ GeV for meson-baryon and meson-meson reactions) color strings are formed and they decay into hadrons according to the Lund string model [٣٤]. There are ٥٥ baryon and ٣٢ meson states as discrete degrees of freedom in the model as well as their antiparticles and explicit isospin projected states with masses up to ٢,٢٥ GeV/c^{٢}. All of these hadronic states can propagate and reinteract in phase space.

٢,٤ ***Differential cross sections***

In the present work, the scattering angles between the outgoing particles are determined by the free [٣٢،٣٥] and medium modified [٣٦،٣٧] differential cross sections. The corresponding expressions are given below.

- ***<u>Free space</u>***

For NN-*elastic* and inelastic scattering we used the parametrizations of Cugnon which takes into account p-n and p-p collisions differently. These parametrizations are mainly adopted in many microscopic transport models.

١. pp-*elastic* scattering [٣٢]:

$$\frac{dc_{pp}}{dt} \approx e^{-A(s)|t|}, \qquad (٢,١٧)$$

where

$$A(s) = ٦ \times \frac{\left(٣.٦٥\left(\sqrt{s} - ١.٨٧٦٦\right)\right)^{٦}}{١ + \left(٣.٦٥\left(\sqrt{s} - ١.٨٧٦٦\right)\right)^{٦}}.$$

(٢,١٨)

٢. pn-*elastic* scattering [٣٥]:

$$\frac{dc_{\ pn}}{dt} \approx e^{B_{pn}t} + \alpha e^{B_{pn}u},$$

(٢,١٩)

the Mandelstam variables *s*, *t* and *u* are given by Eqs. (A٩), (A٢٩) and (A٣٠), respectively (see appendix A). The coefficients B_{pn} and α are given by

$$B_{pn} = ٣.٦٨ + ٠.٧٦ p_{lap},$$

$$\alpha = \left(٠.٨/p_{lap}\right)^{٢}, \qquad (٢,٢٠)$$

where p_{lap} is the incident laboratory momentum in GeV.

٣. $NN - N\Delta$ inelastic scattering [٣٥]:

$$\frac{dc_{\ NN \to N\Delta}}{dt} \approx e^{B_{in}t} + e^{-B_{in}t},$$

(٢,٢١)

with

$$B_{in} = ٥.٢٨٧\left[١ + \exp\left(\frac{p_{lab} - ١.٣}{٠.٠٥}\right)\right]^{-١}. \qquad (٢,٢٢)$$

➢ ***<u>Medium modified</u>***

In the present work we adopt the analytical expressions of NN-*elastic* and inelastic medium modified differential cross sections derived from the collision term of the RBUU equation. The starting point of model is the Lagrangian of an interacting many body system of baryons and mesons. The latter determines the interaction between baryons in terms of nucleon–meson coupling vertices. It is shown that the calculation results can reproduce the experimental free $NN-\ NN$ and $NN-\ N\Delta$ cross sections. Furthermore, the in-medium cross sections are shown to have an obvious dependence on the effective mass and density.

١. NN-*elastic* scattering [٣٦]:

$$\frac{d\sigma_{NN\rightarrow NN}(s,t)}{dt}=\frac{١}{(٢\pi)^{٢}s}\left[\frac{\left(g_{NN}^{\sigma}\right)^{٤}}{٢\left(t-m_{\sigma}^{٢}\right)^{٢}}\left(t-٤m^{*٢}\right)^{٢}+\frac{\left(g_{NN}^{w}\right)^{٤}}{٢\left(t-m_{w}^{٢}\right)^{٢}}\left(٢s^{٢}\right.\right.$$

$$\left.+٢st+t^{٢}-٨m^{*٢}s+٨m^{*٤}\right)+\frac{٢٤\left(g_{NN}^{\pi}\right)^{٤}}{\left(t-m_{\pi}^{٢}\right)^{٢}}m^{*٤}t^{٢}$$

$$- \frac{٤\left(g_{NN}^{\sigma} g_{NN}^{w}\right)^{٢}}{\left(t - m_{\sigma}^{٢}\right)\left(t - m_{w}^{٢}\right)} \times \left(٢s + t - ٤m^{*٢}\right) m^{*٢} \Bigg].$$

(٢,٢٣)

Where $t \neq m_{\pi}, m_{\alpha}$, and m_{c}. g's and m^{*} are the coupling strengths and in-medium mass, respectively. The in-medium single particle energy is given by

$$E^{*}(p) = \sqrt{p^{٢} + m^{*٢}} .$$

(٢,٢٤)

The formula for the differential cross section of in-medium NN-*elastic* scattering is extended to all elementary hh collisions (except for the $NN - N\Delta$ process) by the replacement [١٧],

$$s \rightarrow s - \left(m_{١}^{*} + m_{٢}^{*}\right)^{٢} + ٤m^{*٢},$$

(٢,٢٥)

where $m_{١}^{*}$ and $m_{٢}^{*}$ denote the effective masses of the incoming hadrons.

٢. $NN - N\Delta$ [٣٧]:

$$\frac{d\sigma_{NN\to N\Delta}(s,t)}{dt}=\frac{٨}{(٢\pi)^{٢}s}\left(g^{\pi}_{NN}\right)^{٢}\left(g^{\pi}_{\Delta N}\right)^{٢}\left[\frac{\left(s-m^{*٢}-m^{*٢}_{\Delta}\right)^{٢}-٤m^{*٢}m^{*٢}_{\Delta}}{s\left(s-٤m^{*٢}\right)}\right]^{١/٢}$$

$$\times[D(s,t)+E(s,t)], \qquad (٢,٢٦)$$

with the ***direct*** term,

$$D(s,t)=-\frac{m^{*٢}t\left[\left(m^{*}_{\Delta}+m^{*}\right)^{٢}-t\right]^{٢}\left[\left(m^{*}_{\Delta}-m^{*}\right)^{٢}-t\right]}{٦m^{*٢}_{\Delta}\left(t-m^{٢}_{\pi}\right)^{٢}}, \qquad (٢,٢٧)$$

and the ***exchange*** term,

$$E(s,t)=-\frac{m^{*٢}}{١٢m^{*٢}_{\Delta}\left(t-m^{٢}_{\pi}\right)\left(u-m^{٢}_{\pi}\right)}\sum_{i=١}^{٦}E_i\ ,$$

(٢,٢٨)

where

$$E_{١}=m^{*٢}_{\Delta}[(٨s-٣t)m^{*٢}t-٢(s+٣t)m^{*٤}+٣m^{*٦}-٢s^{٢}t+٢t^{٣}],$$

$$E_{٢}=m^{*٣}_{\Delta}m^{*}[(٢s+t)t-٢(s+t)m^{*٢}+٦m^{*٤}],$$

$$E_{٣}=m^{*}_{\Delta}m^{*}[(٢s-t)m^{*٢}t+(s+٣t)m^{*٤}+(s+t)st-٣m^{*٦}],$$

$$E_{٤}=m^{*٥}_{\Delta}m^{*}[s-t-٣m^{*٢}]+m^{*٤}_{\Delta}[(s-٣t)m^{*٢}+٢st-t^{٢}],$$

$$E_{٥}=(s+٩t)m^{*٦}_{\Delta}+(s+٦t)(s+t)m^{*٢}t-(s+٢t)m^{*٤}t,$$

$$E_{٦}=-m^{*٦}_{\Delta}m^{*٢}-m^{*٨}-t^{٢}(s+t)^{٢},$$

(٢,٢٩)

with $g^{\pi}_{\Delta N}$ =١٥.٦٣. The definition of s is the same as in Eqs. (A٩)

	g_c	g_a	Λ_c (MeV)	Λ_a (MeV)	Λ_π (MeV)
Set A	٦,٩	٧,٥٤	٧٠٤,٣٦	٩٥٠,٠٧	٥٤٥
Set B	٩,٤	١٠,٩٥	[١٩] ١٢٠٠	٨٠٨,٢٩	٥١٠
Set C	٧,٩٣٧	٦,٦٩٦	١١٨٧,٤١	٨٢٣,٦٤	٥٨٥

and (٢,٢٥). While t and u are defined by Eqs. (A٣١) and (A٣٢).

TABLE ١.

Coupling strengths and cut-off masses. m_c = ٥٥٠ *MeV,* m_a = ٧٨٣ *MeV, and* m_π = ١٣٨ *MeV are used for all cases.*

The effects stemming from the finite size of hadrons and a part of the short range correlation is taken into account in Eqs. (٢,٢٣) and (٢,٢٦) by introducing a phenomenological form factor at each vertex. For the baryon-baryon-meson vertex the common form,

$$F_{BBM} = \frac{\Lambda_M^{٢}}{\Lambda_M^{٢} - t}, \qquad (٢,٣٠)$$

is used, where Λ_A is the cut-off mass of the meson A.

In Table ١, the values of coupling strengths g's and cut off masses Λ's are given for three parameter sets. These parameter sets have been investigated by Mao [٣٦], using RBUU model. In Ref. [٣٦], calculations are compared to the experimental results of the

optical potential [٣٨] and mean free path (MFP) [٣٩]. It is found that, the optical potential calculated with constant coupling strengths deviate widely from experimental data, and a better reproduction is obtained if the momentum (density) dependence of the coupling strengths (Eq. (٢،٣٠)) is taken into account. It is also demonstrated that set A and set C can reproduce the energy dependence of optical potential equally well as set B, but they are not as good as set B in reproducing the experimental data of MFP. Thus, only the momentum dependent coupling strength of parameter set B will be adopted in the present work.

Since UrQMD only uses free cross sections and free on-shell particles the effective in-medium quantities E^*, m^*, and m^*_Δ are replaced by the free quantities E, m, and $\langle m_\Delta \rangle$ in actual calculations.

As will be discussed in chapter ٣ it is not enough to define the medium modified differential cross section for the $NN - N\Delta$ reaction but we also need to use an appropriate mass distribution for the Δ- resonance, $\langle m_\Delta \rangle$. We choose [٤٠]

$$\langle m_\Delta \rangle = \frac{\int_{m_N + m_\pi}^{\sqrt{s} - m_N} f(m_\Delta) m_\Delta dm_\Delta}{\int_{m_N + m_\pi}^{\sqrt{s} - m_N} f(m_\Delta) dm_\Delta},$$

(٢,٣١)

where

$$f(m_\Delta) = \frac{١}{\pi} \frac{\Gamma_٠/٢}{(\Gamma_٠/٢)^٢ + (m_\Delta - ١٢٣٢)^٢}, \quad ٢١$$

(٢,٣٢)

Inserting Eq. (٢,٣٢) into Eq. (٢,٣١) one can obtain,

$$\langle m_\Delta \rangle = ١٢٣٢ + (\arctan Z_+ - \arctan Z_-)^{-١} \frac{\Gamma_٠}{٤} \ln\left(\frac{١ + Z_+^٢}{١ + Z_-^٢} \right), \qquad (٢,٣٣)$$

where

$$Z_+ = \left(\sqrt{s} - m_N - ١٢٣٢\right)(٢/\Gamma_٠),$$
$$Z_- = (m_N + m_\pi - ١٢٣٢)(٢/\Gamma_٠),$$

(٢,٣٤)

with $\Gamma_٠ = ١١٠$ MeV. It is shown in Ref. [٤٠] that a successful reproduction of the empirical free NN-*inelastic* cross section can be realized using the mass distribution of Eq. (٢,٣٣). The dependence

of $\langle m_\Delta \rangle$ on s is depicted in Fig. ٢,١. One can find that, as the total energy s increases the $\langle m_\Delta \rangle$ increases so rapidly up to a kinetic energy ١,٢ GeV ($\sqrt{s} \approx$ ٢,١٤ GeV). After reaching the resonance mass $\langle m_\Delta \rangle =$ ١٢٣٢ MeV, the $\langle m_\Delta \rangle$ increases very slowly with the increasing of s.

It is worth stressing that the free and medium modified differential cross sections are used to determine the scattering angles between the outgoing particles in elementary hh collisions but not for the corresponding total cross sections.

The analytical solutions of scattering angles ($\cos\theta$) for the free differential cross sections are derived in appendix B. While in appendix C the corresponding numerical solutions of medium modified differential cross sections are given.

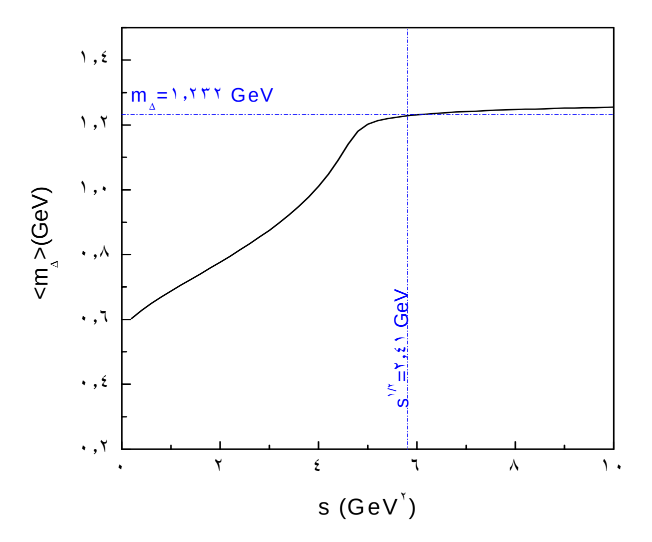


FIG. ٢,١. The delta mass distribution $\langle m_\Delta \rangle$ as a function of the total energy (s) of two particle system in free space.

٢,٥ ***Mean field interaction***

On the basis of quantum٢٤molecular dynamics, potential interactions are enforced for the scattered nucleons. The single particle wave function of each nucleon is represented by a Gaussian wave packet, having the phase-space centroid parameters of $\vec{R}_i$ and $\vec{p}_i$ for the *i*th nucleon. The total wave function is assumed to be a product wave function of nucleon Gaussian wave packet. The equation of motion for their centroids ($\vec{R}_i$ and $\vec{P}_i$) is given by

$$\frac{dR_j}{dt} = \frac{\partial H}{\partial P_j}, \qquad \frac{dP_j}{dt} = -\frac{\partial H}{\partial R_j}.$$

(٢,٣٥)

The Hamiltonian *H* consists of the kinetic energy and the effective interaction energy

$$H = T + V,$$

$$T = \sum_j \left[\left(p_j^{٢} + m_j^{٢} \right)^{١/٢} - m_j \right],$$

$$V = V_{Skyrme} + V_{Yukawa} + V_{Coulomb} + V_{Pauli}. \qquad (٢,٣٦)$$

In this interaction energy, the following terms are included:

Skyrme-type density dependent interaction (V_{Skyrme}), long range Yukawa potential (V_{Yukawa}), Coulomb potential between protons $(V_{Coulomb})$, and the Pauli potential (V_{Pauli}).

The Skyrme-type density dependent interaction consists of a sum of two- and a three- body interaction terms. The two-body term, which has a linear density dependence models the long range attractive component of the NN interaction, whereas the three-body term is responsible for the short range repulsive part of the interaction.

The Pauli potential is introduced for the sake of simulating fermionic properties in semiclassical way. This phenomenological potential prohibits nucleons of same spin and isospin from coming close to each other in the phase space.

The form of each term is given by

$$V_{Skyrme} = \frac{t_{١}}{٢\rho_{٠}} \sum_{i=١}^{A} \sum_{\substack{k=١ \\ k \neq i}}^{A} \tilde{\rho}_{ik} + \frac{t_{\gamma}}{(\gamma+١)\rho_{٠}^{\gamma}} \sum_{i=١}^{A} \left(\sum_{\substack{k=١ \\ k \neq i}}^{A} \tilde{\rho}_{ik} \right)^{\gamma} ,$$

$$V_{Yukawa} = \frac{V_{٠}^{Yuk}}{٢} \sum_{i=١}^{A} \sum_{\substack{k=١ \\ k \neq i}}^{A} \frac{١}{٢\vec{r}_{ik}} \exp\left(\frac{١}{٤\alpha\gamma_{Y}^{٢}} \right) \times \left\{ e^{-\vec{r}_{ik}/\gamma_{Y}} \left[١ - erf\left(\frac{١}{٢\gamma_{Y}\sqrt{\alpha}} - \sqrt{\alpha}\vec{r}_{ik} \right) \right] \right.$$

$$\left. - e^{-\vec{r}_{ik}/\gamma_{Y}} \left[١ - erf\left(\frac{١}{٢\gamma_{Y}\sqrt{\alpha}} + \sqrt{\alpha}\vec{r}_{ik} \right) \right] \right\},$$

$$V_{Coulomb} = \frac{١}{٢} e^{٢} \sum_{i=١}^{A} \sum_{\substack{j=١ \\ j \neq i}}^{A} \frac{١}{\vec{r}_{ij}} erf\left(\sqrt{\alpha}\vec{r}_{ij} \right) ,$$

$$V_{Pauli} = \frac{١}{٢} V_{٠}^{P} \left(\frac{\hbar}{p_{٠}q_{٠}} \right)^{٣} \left(١ + \frac{١}{٢\alpha q_{٠}^{٢}} \right)^{-٣/٢} \times \sum_{i=١}^{A} \sum_{\substack{k=١ \\ k \neq i}}^{A} \exp\left(\frac{-\alpha r_{ik}^{٢}}{٢\alpha q_{٠}^{٢} + ١} - \frac{p_{ik}^{٢}}{٢p_{٠}^{٢}} \right) \delta_{\tau_i \tau_k} \delta_{\upsilon_i \upsilon_k} ,$$

(٢,٣٧)

where $\vec{r}_{ik} = \vec{R}_i - \vec{R}_k$, $\vec{p}_{ik} = \vec{P}_i - \vec{P}_k$, τ_i and υ_i denote the spin-isospin index of nucleon (i), and the 'interaction density'

$$\tilde{\rho}_{ik} = (\alpha/\pi)^{٣/٢} e^{-\alpha\left(\vec{R}_i - \vec{R}_k\right)^{٢}} .$$

The summation runs over all projectile and target nucleons, $\rho_{٠}$ = ٠.١٦٨ fm$^{-٣}$ is the normal nuclear density, and erf denotes the error function. The values of the parameters appearing in Eq. (٢,٣٧)

are given in Table ٢ [١٧], which correspond to the medium equation of states with compressibility $K =$ ٢٦٠,٢٢ MeV.

TABLE ٢. *Parameters of the interaction potential.*

a (fm^-٢)	$t_{١}$ (MeV)	t_{γ} (MeV)	γ	٢٧$V_{٠}^{Yuk}$ (MeV fm)	γ_{Y} (fm)	$V_{٠}^{P}$ (MeV)	$q_{٠}$ (fm)	$p_{٠}$ (MeV/c)
٠,١١٥٢	-٨٤,٥	١٨٨,٢	١,٤٦	-٨٥,١	١	٩٩,٥	٣	١٢٠

The UrQMD calculation is carried out up to a time scale referred to as the transition time t_{tr}, the time at which the UrQMD is stopped to give way to evaporation. We have selected t_{tr} to be ١٠٠ fm/c, because this value was enough to obtain stable neutron spectra from the (p, xn) reaction against a change of t_{tr} as shown in Ref. [٤١]. At $t_{tr} =$ ١٠٠ fm/c, the position of each nucleon is used to calculate the distribution of mass and charge numbers (referred to as 'prefragments'). In determining the mass and charge numbers of the prefragments, the minimum spanning tree method [٤٢] is employed, a prefragment is formed if the centroide of their wave packets have a spatial distances $d_{\circ}$ ؛ ٣ fm. The prefragments thus identified are

then Lorentz boosted into their rest frames to evaluate their excitation energies. The excitation energy E^{*} of hot prefragments is calculated as the difference between the binding energy of the hot prefragments E' and the binding energies of these prefragments in their ground state $E^{\circ}, E^{*} = E' - E^{\circ}$. When the prefragment is in the excited state, the statistical decay via $n, p, d, t,$ ٣$He,$ and α emissions is consi-dered based on the Weisskopf approximation [٤٣].

In the numerical calculations, the UrQMD (version ١,٢) is run in two modes, the cascade mode (UrQMD/C) and the one that includes the mean field effect (UrQMD/M).

In addition, the UrQMD predictions are compared using free space and medium modified differential cross sections for NN-*elastic* and $NN - N\Delta$ processes.

In the next chapter we denote the improvements established using the medium modified NN-*elastic* and $NN - N\Delta$ differential cross section as well as changes to the Δ- mass distribution in the UrQMD/C and UrQMD/M as ' UrQMD/CM' and ' UrQMD/MM',

respectively. In this work, the default UrQMD parameters are selected, and no adjustment is attempted.

Chapter ٣

Results and discussion

In this section, we display the predictions of the UrQMD model (coupled with free and medium modified differential cross sections) along with the recent measurements [٢٠] of double differential neutron production cross sections as a function of neutron kinetic energy (E_n) at ٠° and ١٠° for $p+{}^{٢٧}Al$, ${}^{٥٦}Fe$, and ${}^{٩١}Zr$ at ١,٢ GeV, in which both the quasielastic and inelastic peaks are prominent. A full comparison between the UrQMD calculations, without the $NN-N\Delta$ medium modified differential cross sections, and the data at various angles can be found in Ref. [٤١] for $p+{}^{٢٧}Al$, ${}^{٥٦}Fe$, and ${}^{٩١}Zr$ at ١,٢ GeV.

In this chapter we discuss the validity of Eq. (٢,٢٦) for binary collision $NN-N\Delta$ in Sec. (٣,١). Experimental and theoretical uncertainties are reported in Sec. (٣,٢). In Secs. (٣,٣) and (٣,٤) the

influence of in-medium and mean field effects, respectively, is illustrated. The comparison with other models is given in Sec. (٣,٥). We investigate the influence of the reverse process $N\Delta - NN$ in Sec. (٣,٦).

٣,١ ***Validity of formulation of medium modified NN-inelastic scattering***

Let us first check the validity of Eq. (٢,٢٦) for binary collision $NN - N\Delta$. Figure ٣,١ displays the angular distribution of the neutron calculated using Eq. (٢,٢٦) for the reaction $pp \to np\pi^+$ at $E_{lab} =$ ٠.٩٧ GeV as compared with experiment [٤٤]. Note that ٩٠٪ of this reaction goes through $pp \to n + \Delta^{++} \to n + p + \pi^+$ and only ١٠٪ through $pp \to p + \Delta^+ \to p + n + \pi^+$. The results of Eq. (٢,٢٦) are displayed in Fig. ٣,١(a) for different values of $\langle m_\Delta \rangle$. When $\langle m_\Delta \rangle = m_N + m_\pi$ the best fit to the experimental data can be obtained [see Fig. ٣,١(b)]. The contributions of the direct term, Eq. (٢,٢٧), and the exchange term, Eq. (٢,٢٨), are also shown in Fig. ٣,١(b). One can easily find that the observed neutron angular distribution

can be reached if only the exchange term is taken into account. The underestimation of the observed angular distribution around $\theta = ٩٠^\circ$ is of no importance as far as the numerical simulations are concerned.

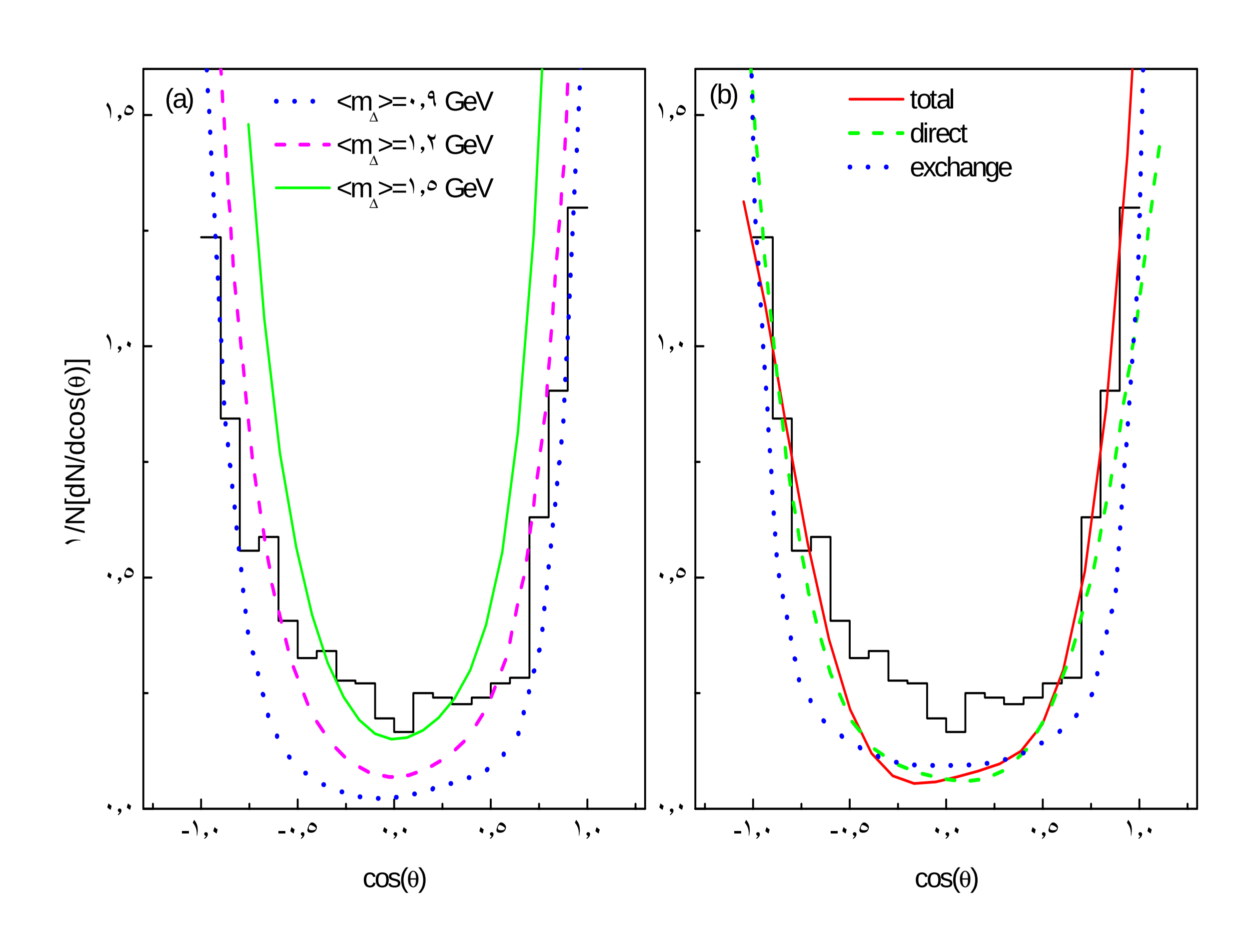


FIG. ٣,١. The angular distribution of the neutron calculated using Eq. (٢,٢٦) for the reaction $p + p \rightarrow n + \Delta^{++} \rightarrow n + p + \pi^{+}$ at ٠,٩٧ GeV. (a) Shows the calculations with

different delta mass distributions. (b) The contributions of the direct and exchange terms. The solid histograms show the data from Ref. [٤٤].

٣,٢ ***Experimental and theoretical uncertainties***

The measured energy spectra (see Figs. ٣,٢ and ٣,٤) are characterized by a narrow peak at a kinetic energy near that of the beam energy and a broad peak at lower energy centered around ٨٧٣ MeV and ٧٦٠ MeV at ٠° and ١٠°, respectively.

The upper peak (denoted as "***quasielastic peak***") is due to a single (p,n) elastic scattering in the forward direction and shifted toward large energy losses compared to the quasi-free kinematics, by ٢٠-٣٠ MeV, more or less independently of the target mass. The lower peak (denoted as "***quasi-inelastic peak***" is about ٤٠٠ MeV wide and is thought to be due to Δ- resonance excitation: $NN - N\Delta$ (a single (p,n) inelastic scattering with multiple scattering contributions).

The thickness of the targets also induces some distortion in the neutron double differential spectra. The effect of the target thickness on the neutron spectra results in a depopulation of the intermediate energy part of the spectra (between ٢٠٠ and ٦٠٠ MeV) and a ٣٠ MeV downward shift of the location of the quasielastic peak [٢٠]. Calculations using LAHET high energy transport code system [٤٥] were performed in Ref. [٢٠] for targets with actual geometry and

infinitely thin one in order to asses the order of magnitude of the depopulation. It is shown that the difference is very small for the Pb target and becomes larger as both the target mass number and angle decrease. The UrQMD predictions shown here do not include these corrections.

Below, we are going to investigate these two peaks by employing the UrQMD model with different (free and medium modified) angular distributions. We performed ٢٠٠٠٠ simulations at various impact parameters from ٠ to $R+$ ٠.٥ fm, where R is the target radius given by Eq. (٢,٢). In order to have sufficient statistics, calculations were done for angular bins of ٣,٥ and ٥ at ٠$^{\circ}$ and ١٠$^{\circ}$, respectively.

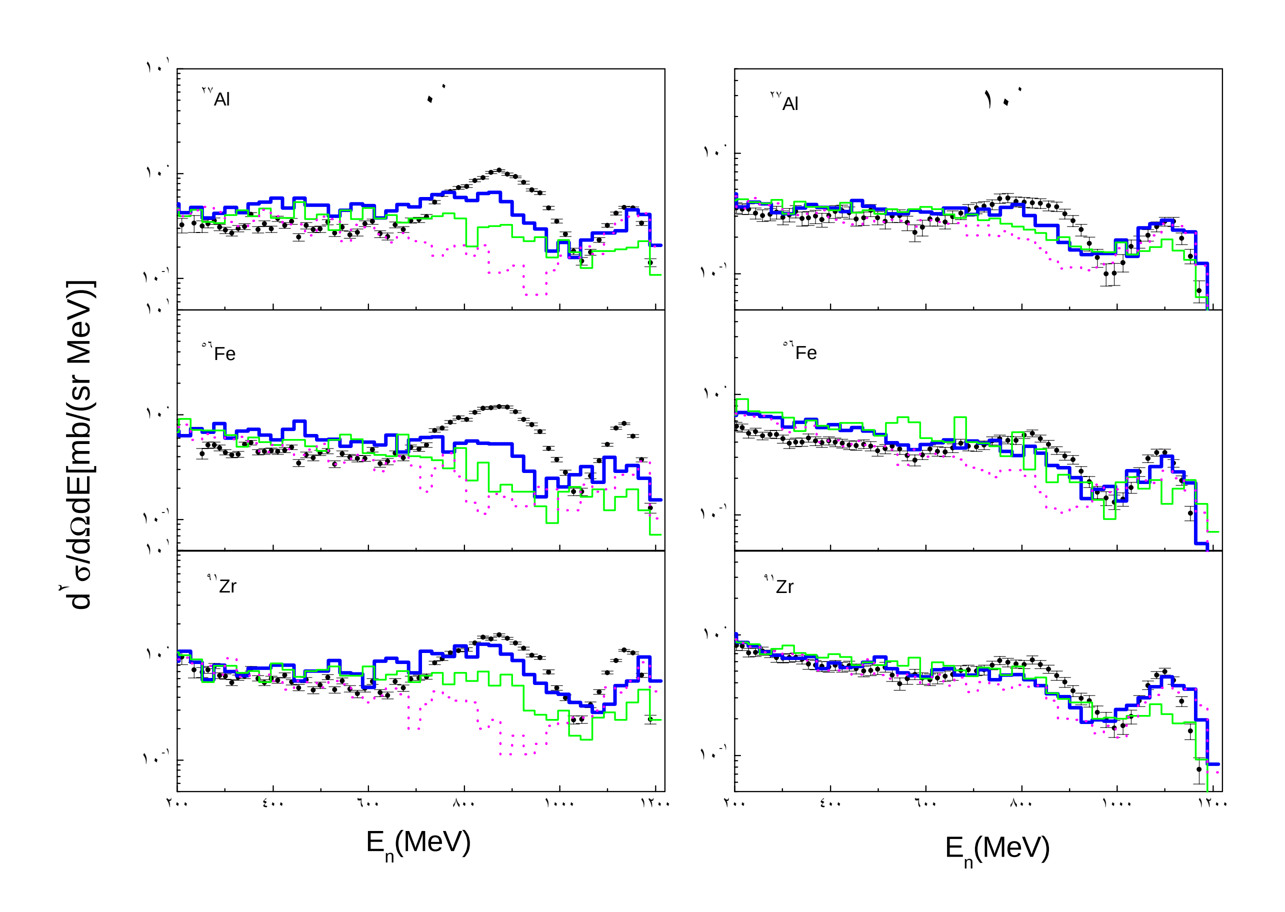


FIG. ٣,٢. Neutron energy spectra at ٠° (left panels) and ١٠°(right panels) from ١,٢ GeV proton interactions with targets of (from the bottom): ٩١Zr, ٥٦Fe, and ٢٧Al. The bold

solid histograms denote the UrQMD/C calculations with the simulated mass distribution of Eq. (٢,٣٣), while the dotted histograms are those with $\langle m_\Delta \rangle = m_N + m_\pi$. The thin solid histograms represent the UrQMD/C calculations with free parametrizations. The experimental data (solid circles with error bars) are taken from Ref. [٢٠]

٣,٣ ***<u>Influence of in-medium angular distribution</u>***

In Fig. ٣,٢ we plot the double differential cross sections of the neutron as a function of E_n at ٠° (left panels) and ١٠° (right panels) for the reactions under study. The solid circles with error bars represent the experimental data. The histograms denote the results of the UrQMD/C. In the same figure, we plot the results of UrQMD/C with different choices of $\langle m_\Delta \rangle$. The dotted histograms are the results obtained with $\langle m_\Delta \rangle = m_N + m_\pi$, while the bold solid histograms are those with the simulated mass distribution of Eq. (٢,٣٣). The former case corresponds to neutrons following the quasi-free pion production from reactions like $pn \rightarrow pn\pi^\circ$, $pn \rightarrow nn\pi^+$, and $pp \rightarrow np\pi^+$, and the latter to the Δ- resonance excitation; $NN - N\Delta$. As one can see the quasi-inelastic peaks at ٠° and ١٠° are

predominately determined by the mass distribution of the Δ- resonance.

The quasielastic peaks at ٠° are underestimated by the UrQMD/C calculations [using the centroid mass of Eq. (٢،٣٣)] for the reactions under study, although they are better reproduced for $p+{}^{٢٧}Al$, and ${}^{٩١}Zr$. In contrast, the intensity of the quasielastic peaks are reproduced at ١٠°. ٣٦

Let us next investigate the influence of in-medium correction on the quasielastic and inelastic peaks for the reactions under study. The in-medium correction is defined by the difference between the observables for medium modified differential cross sections, Eqs. (٢،٢٣) and (٢،٢٦), and for free ones, Eqs. (٢،١٧), (٢،١٩), and (٢،٢١), in the nuclear medium. For the latter case we use the same parametrizations presented in Ref. [٣٥], with which a successful reproduction of the empirical free NN- differential cross sections are obtained. The predicted differential cross sections at several energies for $pn - pn$ and $NN - NN\pi$ are displayed in Figs. ٣،٣. We see that, although the angular distributions for free parametrizations are more forward peaked (cf. Figs. ٣،٣), the

UrQMD/C calculations in conjunction with the medium modified parametrizations lead to an enhancement of the quasi-inelastic (at ٠°) and elastic(at ١٠°) peaks (see Fig. ٣,٢). On the other hand, the quasi-inelastic peaks at ١٠° are rather insensitive to different parametrizations.

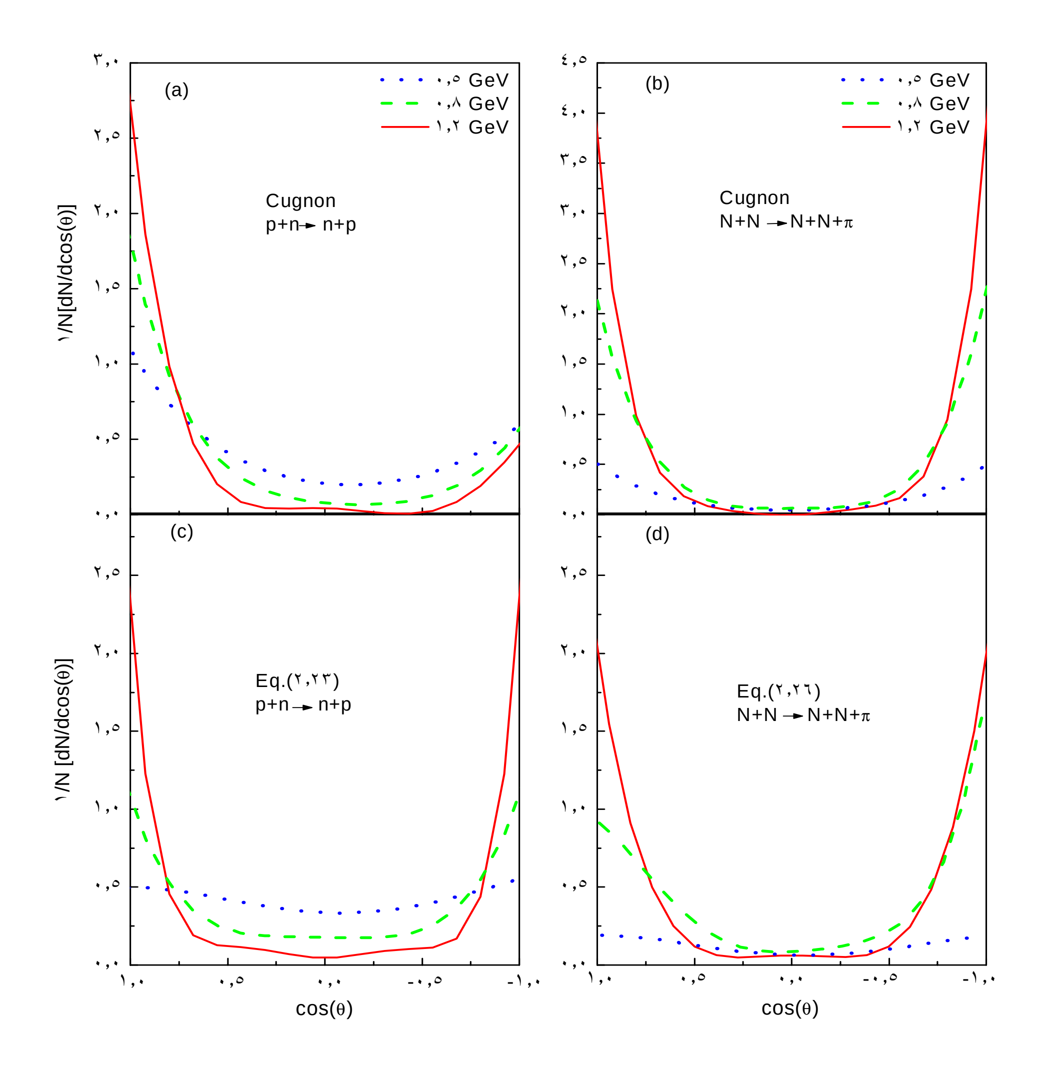


FIG. ٣,٣. The predicted angular distributions of neutrons evaluated at different laboratory energies for $p+n \to p+n$ (left panels) and $N+N \to N+N+\pi$ (right panels). (a) and (b) are the results calculated by the free Cugnon parametrizations, Eqs.

(٢,١٧), (٢,١٩), and (٢,٢١), while (c) and (d) are those calculated by Eqs. (٢,٢٣) and (٢,٢٦), respectively.

٣,٤ ***Influence of the mean field***

In order to study the influence of the mean field on the quasielastic and inelastic peaks, we compare in Fig. ٣,٤ both the UrQMD/CM and UrQMD/MM results with the experimental data for the reactions under study. We find that the mean field effect is most dramatic in the quasielastic region whereas it is less dramatic in the quasi-inelastic region. Both the intensity and the location of quasielastic peaks at ٠° are now better reproduced by the UrQMD/MM calculations. For the location, we neglected ٣٠ MeV downward shift of the peak location, which arises from the thickness of the target: Taking into account this shift would yield an even better agreement with the data. In contrast, at ١٠° the quasielastic peaks are getting broader in comparison to the data with increasing the mass number of the target nucleus. On the other hand, the

broadening of the quasi-inelastic peaks is satisfactorily reproduced by the UrQMD/MM calculations at ٠° and ١٠°, as the mass number of the target increases.



FIG. ٣,٤. Same as Fig. ٣,٢, but here the bold solid and dotted histograms denote the UrQMD/MM and UrQMD/CM calculations, respectively. The thin solid histograms denote the UrQMD/MM calculations that includes the $N + \Delta - N + N$ process.

From Fig. ٣،٤ one notices that the neutron spectra below the quasi-inelastic peak are overestimated by the UrQMD/MM calculations at ٠°, and to a lesser extent at ١٠°. Part of the overestimation may be due to the neglect of the finite thickness of the target by the UrQMD/MM calculations.

٣,٥ ***Comparison with other models***

In Ref. [٢١] using the intranuclear cascade (INC) model, several effects are used to investigate the quasielastic and inelastic peaks for $p+{}^{٢٠٨}Pb$ at ٨٠٠ MeV. These effects include in-medium cross sections, refraction at the nuclear surface, stopping time, Pauli blocking and a diffuse nuclear surface. In all the effects studied, the only one which increase the intensity of the quasielastic and inelastic peaks is the introduction of a diffuse nuclear surface.

Recently [٢٢], a new version of the INC model (INCL٤) is proposed, which accommodates a diffuse nuclear surface, for the description of the reactions under study. It is shown that the width of the quasielastic and inelastic peaks are underestimated by a factor of ٢ or so. This may suggest that it is very important to take into account the mean field as well as the medium modified $NN - NN$ and $NN - N\Delta$ differential cross sections for the description of the quasielastic and inelastic peaks in proton induced reactions.

٣,٦ ***Influence of the reverse process***

In Fig. ٣,٤ we additionally investigate the influence of the reverse process $N\Delta - NN$ on the quasi-inelastic and elastic peaks for the reactions under study. It is assumed, as in Ref. [٣٧], that the angular distributions of $N\Delta - NN$ and $NN - N\Delta$ are similar. We show that the implementation of this channel in the UrQMD/MM calculation improves the intensity of both the quasi-inelastic (at ٠° and ١٠°) and elastic (at ٠°) peaks (see thin histograms in Fig. ٣,٤) but leads to a broadening of the quasielastic peaks, especially at ١٠°.

This also indicates that the delta degrees of freedom survive in the nuclear medium.

Chapter ٤

٤٢

Summary and Outlook

Two distinct contributions of the nuclear medium are investigated in the UrQMD framework, referred to as "in-medium" and "mean field" effects. The former includes modification to the NN-*elastic* and $NN - N\Delta$ angular distributions as well as changes to the Δ- mass distribution. The latter incorporates interaction potential between nucleons. The in-medium effect is defined as the difference between the observables for the medium modified angular distributions and for free ones. The mean field effect is given as the difference between the UrQMD/MM and UrQMD/CM on the observables, that is when the two running UrQMD modes are

supplemented with the medium modified angular distributions as well as changes to the Δ- mass distribution.

The influence of these two effects on the quasielastic and inelastic peaks of neutrons at ٠° and ١٠° are studied for $p+{}^{٢٧}Al$, ${}^{٥٦}Fe$, and ${}^{٩١}Zr$ at ١,٢ GeV. From the calculation results we can get the following conclusions:

١) The quasi-inelastic peak is predominately determined by the mass distribution of the intermediate excited delta resonance:

The delta degrees of freedom survive in the nuclear medium.

٢) The in-medium effect plays an important role for enhancing the quasi-inelastic peak.

٣) The mean field effect enhances both peaks and becomes important in the quasielastic region.

٤) A better reproduction of the two peaks is obtained with UrQMD/MM than with UrQMD/C using free angular distributions.

٥) Including the $N\Delta - NN$ process in the UrQMD/MM calculations improves the description of the quasi-inelastic peak.

٦) The neutron spectra below the quasi-inelastic peak are satisfactorily reproduced by the UrQMD/MM when the effect of target thickness plays a minor role.

Thus, the quality of the results presented here and elsewhere [٤١] (see Refs. [٢١،٢٢] for a comparison) induces us to believe that the UrQMD approach is much more appropriate for taking proper account of the collision process. However, further possible improvements of the UrQMD model at intermediate energies are still needed. These include the following:

i. As we do for the angular distributions, one should use medium modified cross sections for $NN-NN$ and $NN-N\Delta$ processes.

ii. Besides the inclusion of the delta in the collision part and in the delta mass distribution it also has to be included in the mean field part of the theory.

iii. More realistic spatial and momentum densities should be used instead of the crude box approximation.

iv. A self-consistent minimization of the energy of the initial nuclei should be implemented instead of the normal packing procedure.

Appendix A

In this appendix we will determine the Mandelstam variables *s*, *t* and *u* which are frequently used in this work.

Consider the relativistic kinematics of two body reactions:

$$A + B \to C + D. \qquad \text{(A١)}$$

We denote respectively by m_A, m_B, m_C, and m_D, the rest masses of the particles *A*, *B*, *C*, and *D*, while their four momenta are called p_A, p_B, p_C, and p_D.

Note that the four momenta $p \equiv (E/c, \vec{p})$ are calculated according to[*]

$$p^٢ = E^٢/c^٢ - |\vec{p}|^٢ = m^٢c^٢ . \qquad (A٢)$$

In what follows, and in order to simplify the formulae, we shall frequently choose units such that $c = ١$.

Conservation of energy and momentum gives

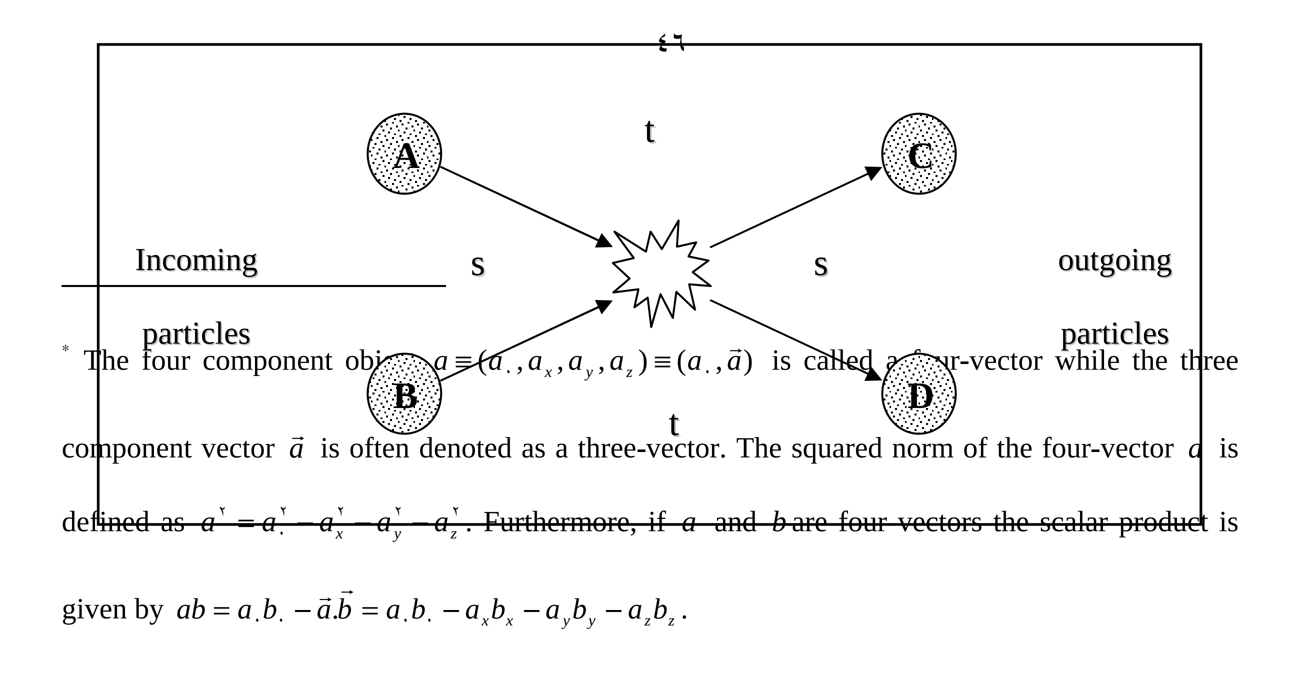
$$p_A + p_B = p_C + p_D ,$$

(A٣) with the additional constraints (see Eq. A٢)

$$p_A^٢ = m_A^٢, \quad p_B^٢ = m_B^٢, \quad p_C^٢ = m_C^٢, \quad p_D^٢ = m_D^٢, \qquad (A٤)$$

which are called "mass-shell conditions".

The Lorentz invariant Mandelstam variables are defined by:



[*] The four component object $a \equiv (a_٠, a_x, a_y, a_z) \equiv (a_٠, \vec{a})$ is called a four-vector while the three component vector $\vec{a}$ is often denoted as a three-vector. The squared norm of the four-vector a is defined as $a^٢ = a_٠^٢ - a_x^٢ - a_y^٢ - a_z^٢$. Furthermore, if a and b are four vectors the scalar product is given by $ab = a_٠b_٠ - \vec{a}.\vec{b} = a_٠b_٠ - a_xb_x - a_yb_y - a_zb_z$.

FIG. A١. Schematic representation of Mandelstam variables

$$s = (p_A + p_B)^{٢} = (p_C + p_D)^{٢},$$

(A٥) $$t = (p_A - p_C)^{٢} = (p_B - p_D)^{٢},$$

(A٦) and

$$u = (p_A - p_D)^{٢} = (p_C - p_B)^{٢}. \qquad (A٧)$$

One directly obtains from Eqs. (A٥-A٧) the following linear relation between *s*, *t* and *u*:

$$s + t + u = m_A^{٢} + m_B^{٢} + m_C^{٢} + m_D^{٢}. \qquad (A٨)$$

The quantity *s* is the square of the total energy in the C.M system. *t* and *u* are the squares of four momentum transfers.

In the C.M system, the invariant quantity *s* may be written as,

$$s = (p_A + p_B)^{٢} = (E_A + E_B)^{٢} - (\vec{p}_A + \vec{p}_B)^{٢} = (E_A + E_B)^{٢}.$$

(A٩) Since $\vec{p}_A + \vec{p}_B = ٠$ in the C.M system. The same quantity *s* evaluated in the laboratory system is given by,

$$s = \left(E_A^L + m_B\right)^{٢} - \left(\vec{p}_A^{\,L}\right)^{٢}$$

$$= \left(E_A^L\right)^{٢} + ٢ m_B E_A^L + m_B^{٢} - \left(\vec{p}_A^{\,L}\right)^{٢}, \qquad (A١٠)$$

where E_A^L and $\vec{p}_A^{\,L}$ are the energy and three momentum in the laboratory system.

But

$$\left(E_A^L\right)^{٢} - \left(\vec{p}_A^{\,L}\right)^{٢} = m_A^{٢},$$

(A١١) and therefore

$$s = m_A^{٢} + m_B^{٢} + ٢ m_B E_A^L. \qquad (A١٢)$$

Introducing the kinetic energy of the incident particle in the laboratory system

$$T_A^L = E_A^L - m_A, \qquad (A١٣)$$

we also have

$$s = \left(m_A + m_B\right)^{٢} + ٢ m_B T_A^L. \qquad (A١٤)$$

At very high energies, we have

$$\left|\vec{p}_A^{\,L}\right| >> m_A, m_B, \qquad (A١٥)$$

and

$$E_A^L \approx \left|\vec{p}_A^{\,L}\right|, \qquad (A١٦)$$

Therefore Eq. (A١٢) becomes

$$s \approx ٢ m_B \left| \vec{p}_A^{\,L} \right|. \qquad (A١٧)$$

So that the square of the C.M energy grows linearly with the laboratory incident momentum at ultra-relativistic energies.

As an application of these formulae (Eqs. A٩-A١٧), let us consider a colliding beam experiment in which two protons, of mass $m = ٠,٩٤$ GeV, move towards each other with C.M three momenta of magnitude $\left|\vec{p}_A\right| = \left|\vec{p}_B\right| = ٣٠$ GeV/c.

Thus $\qquad E_A = E_B = \sqrt{٠.٨٨ + ٩٠٠}$ GeV ≈ ٣٠ GeV/c,

and the invariant quantity s is given by

$$s = \left(E_A + E_B\right)^٢ \approx ٣٦٠٠ \text{ GeV}^٢.$$

According to Eq. (A١٢) we see that the same value of s would require a laboratory three momentum of magnitude

$$\vec{p}_L = \frac{١}{٢m}\sqrt{s^٢ - ٤m^٢ s} \qquad (A١٨)$$

$$= ١٩١٣.٩ \text{ GeV/c}.$$

This clearly shows the interest of using colliding beam experiments to explore the domain of such high-energy collisions.

From Eq. (A٦) we have

$$t = m_A^٢ + m_C^٢ - ٢\, p_A\, p_C$$

$$= m_A^{٢} + m_C^{٢} - ٢(E_A E_C - \vec{p}_A . \vec{p}_C)$$

$$= m_A^{٢} + m_C^{٢} - ٢E_A E_C + ٢|p_A||p_C|\cos\theta . \qquad \text{(A١٩)}$$

where θ is the scattering angle between $\vec{p}_A$ and $\vec{p}_C$.

Defining the initial momentum

$$\vec{p}_i = \vec{p}_A = - \vec{p}_B ,$$

(A٢٠) and the final momentum

$$\vec{p}_f = \vec{p}_C = - \vec{p}_D , \qquad \text{(A٢١)}$$

we have

$$s = (E_A + E_B)^{٢} = \left(\sqrt{m_A^{٢} + \vec{p}_i^{\,٢}} + \sqrt{m_B^{٢} + \vec{p}_i^{\,٢}}\right)^{٢} . \qquad \text{(A٢٢)}$$

So that

$$\vec{p} = \frac{١}{٢\sqrt{s}} \sqrt{\left(s - m_A^{٢} - m_B^{٢}\right)^{٢} - ٤ m_A^{٢} m_B^{٢}} . \qquad \text{(A٢٣)}$$

Introducing the triangle function

$$\lambda(x, y, z) = (x - y - z)^{٢} - ٤yz = x^{٢} + y^{٢} + z^{٢} - ٢xy - ٢yz - ٢zx , \qquad \text{(A٢٤)}$$

we have

$$\vec{p}_i = \frac{١}{٢\sqrt{s}} \sqrt{\lambda\left(s, m_A^{٢}, m_B^{٢}\right)} . \qquad \text{(A٢٥)}$$

Similarly p_f is given by

$$\vec{p}_f = \frac{١}{٢\sqrt{s}}\sqrt{\lambda\left(s, m_C^٢, m_D^٢\right)}.$$

(A٢٦) From Eq. (A٧) we can also deduce that

$$u = m_A^٢ + m_D^٢ - ٢E_A E_D - ٢|p_i||p_f|\cos\theta$$

$$= m_B^٢ + m_C^٢ - ٢E_B E_C - ٢|p_i||p_f|\cos\theta. \qquad (A٢٧)$$

<u>For an elastic collision:</u>

We have

$$m_A = m_C, \quad m_B = m_D, \quad |p_i| = |p_f| = |p|,$$

and from Eq. (A٢٣)

$$|p|^٢ = \frac{s}{٤} - m^٢, \qquad (A٢٨)$$

where $m_A = m_B = m$.

We thus have

$$t = -٢|p|^٢(١ - \cos\theta)$$

$$= \frac{١}{٢}\left(s - ٤m^٢\right)(\cos\theta - ١), \qquad (A٢٩)$$

while

$$u = -٢|p|^٢(١ + \cos\theta)$$

$$=-\frac{١}{٢}\left(s-٤m^{٢}\right)\left(١-\cos\theta\right). \qquad (A٣٠)$$

<u>For an inelastic collision</u>**:**

We have

$$m_A=m_B=m_C=m,\;\; m_D=m_\Delta,$$

$$t=\frac{١}{٢}\left(٣m^{٢}+m_\Delta^{٢}-s\right)+٢|p||p_{٣}|\cos\theta. \qquad (A٣١)$$

From Eq. (A٨) we obtain,

$$u=٣m^{٢}+m_\Delta^{٢}-s-t,$$

(A٣٢) using Eqs. (A٢٥ and A٢٦) we can express p_i and p_f as:

$$p_i=\frac{١}{٢}\sqrt{s-٤m^{٢}},$$

(A٣٣)

$$p_f=\frac{١}{٢\sqrt{s}}\sqrt{\left(s-m^{٢}-m_\Delta^{٢}\right)^{٢}-٤m^{٢}m_\Delta^{٢}}.$$

(A٣٤)

Appendix B

In this appendix we will generate random deviates of the scattering angle $\cos\theta$ from the free pp, pn-*elastic* and NN-*inelastic* angular distributions using the inverse transform method for random number generator [٤٦].

B.I Inverse transform method

The inverse transform method starts by defining the probability density $p(x)$, where $p(x)dx$ is the probability of generating a number between x and $x+dx$. The probability density $p(x)$ is normalized such that

$$\int_{-\infty}^{\infty} p(x)dx = ١ .\qquad (B١)$$

Let us define a uniform random number (ξ) in the unit interval [٠, ١] with the probability density function

$$p_u(r) = \begin{cases} ١ & ٠ \le r \le ١ \\ ٠ & \textit{otherwise.} \end{cases}$$

(B٢) A variable which satisfies the probability distribution (B٢) is known as a uniform deviate.

The aim of the inverse transform method is to find a relation between x and ξ such that if ξ is distributed according to (B٢), x will be distributed according to $p(x)$. In order to obtain this relation, we compute the integral

$$p(x) = \int_{-\infty}^{x} p(x')dx' .\qquad (B٣)$$

The relation for the value of x corresponding to the value of ξ is

$$p(x) = \xi .\qquad (B٤)$$

which leads to the desired relation

$$x = p^{-١}(\xi) .\qquad (B٥)$$

The sequence of steps associated with the inverse transform method

is to generate a random number ξ and solve (B٥) for the corresponding value of x.

B.II *Free* pp-*elastic angular distribution*

Let us now apply the inverse transform method to the angular distribution for pp-*elastic* scattering

$$f(t) = ce^{-A(s)t}, \qquad (B٦)$$

where c is a normalization constant. A(s) and t are given by Eqs. (٢,١٨) and (A٢٩), respectively.

Make a change of variables to angular coordinates, Eq. (B٦) becomes

$$f(\cos\theta) = ce^{٢A(s)p^{٢}} e^{-٢p^{٢}A(s)\cos\theta}, \qquad (B٧)$$

where p is given by Eq. (A٢٨).

If we substitute (B٧) into (B١) and perform the integration on the interval [-١, ١], we find the normalization constant c:

$$c = A(s)/(e^{٤p^{٢}A(s)} - ١). \qquad (B٨)$$

Substituting (B٧) into (B٣) and perform the integration on the interval [-١, $\cos\theta$], we can write

$$\cos\theta = \frac{١}{٢p^{٢}A(s)}\ln\left(\frac{A(s)\xi}{c}\right) - ١. \qquad (B٩)$$

A function subprogram which generates the scattering angle $\cos\theta$ for pp-*elastic* scattering is then given by

```
    function costh-ppel (s,m)
c   generate cosθ for pp-elastic scattering
c   input: s,m
c   output costh: cos(theta) of theta-angle
c   rndm(-١) is the random number in the interval [٠،١]
    p=٠,٥*sqrt(s-٤,٠*m**٢)
    A_s=٦,٠*(٣,٦٥*(sqrt(s)-١,٨٧٦٦))**٦/(١,٠٥+(٣,٦٥*(sqrt(s)-١,٨٧٦٦))**٦)
    c= A_s/(exp(٤,٠* p**٢* A_s-١,٠))
    costh-ppel =log(A_s*rndm(-١)/c)/(٢,٠* p**٢* A_s)-١,٠
    return
    end
```

B.III ***<u>Free</u>*** **<u>pn-</u>*****<u>elastic angular distribution</u>***

The pn-*elastic* angular distribution is given by

$$f(t) = c\left(e^{B_{pn}t} + \alpha e^{B_{pn}u}\right), \qquad (B١٠)$$

where the parameters entered in Eq. (B١٠) are defined in Eqs. (٢,٢٠), (A٢٩), and (A٣٠). We apply the inverse transform method for each term (similar to Eq. (B٦)) in Eq. (B١٠) independently,

depending on the value of (α) which is to be compared with the random number ξ.

The resulting $\cos\theta$ generator will look something like

```
      function costh-pn (s,m)
c     input:s,m
c     output costh: cos(theta) of theta-angle
      Plap=sqrt(s**٢-٤*m**٢*s)/٢*m         ! Plap deduce from Eq. (A١٨)
      p=٠,٥*sqrt(s-٤,٠*m**٢)
      B_pn=٣,٦٨+٠,٧٦*Plap
      if (Plap.le.٠,٨) Alpha=١,٠
      if (Plap.ge.٠,٨) Alpha=(٠,٨/Plap)**٢
      if (rndm(-١).le. Alpha/(١,٠+ Alpha)) then
c     left or right term                    ٥٧
      A= B_pn/(١,٠+exp(-٤,٠* B_pn* p**٢))/Alpha
      costh-pn=(log(rndm(-١)* B_pn/(A*Alpha)-١,٠))/(٢,٠*p**٢*B_pn)-١,٠
      else
      A= B_pn/(١,٠-exp(-٤,٠* B_pn* p**٢))
      costh-pn=log(rndm(-١)* B_pn/A)/(٢,٠* B_pn* p**٢)-١,٠
      end if
      return
      end
```

B.IV ***Free NN-inelastic angular distribution***

The free angular distribution for $NN - N\Delta$ process is given by

$$f(t)=c\left(e^{B_{in}t}+e^{-B_{in}t}\right). \qquad \text{(B١١)}$$

The dependence of the slope parameter B_{in} on the laboratory momentum is given by Eq. (٢,٢٢).

The corresponding function subprogram is

```
function costh-NNine (s,m)
Plap=sqrt(s**٢-٤*m**٢*s)/٢*m                ! Plap deduce from Eq. (A١٨)
p=٠,٥*sqrt(s-٤,٠*m**٢)
B_in=٥,٢٨٧*(١,٠+exp((Plap-١,٣)/٠,٠٥))**-١
if (rndm(-١).le.٠,٥) then
A=B_in/(١,٠-exp(-٤,٠*B_in*p**٢))
costh-NNine=log(rndm(-١)*B_in/A)/(٢,٠*B_in*p**٢)-١,٠
else
A=B_in/(exp(٤,٠*B_in*p**٢)-١,٠)
costh-NNine=log(rndm(-١)*B_in/A)/(٢,٠*B_in*p**٢)-١,٠
end if
return
end
```

Appendix C

Here we derive the scattering angles $(\cos\theta)$ according to the medium modified $NN - NN$ and $NN - N\Delta$ angular distributions given by equations (٢,٢٣) and (٢,٢٦), respectively.

The calculations performed here use the inverse transform method:

$$\xi = \frac{\int_{٠}^{-t} \frac{d\sigma}{dt} dt}{\int_{٠}^{-t_{\max}} \frac{d\sigma}{dt} dt}, \qquad \text{(C١)}$$

where $\frac{dc}{dt}$ is the differential cross section. t and t_{max} are defined below.

The bisection method (C.IV) is used for finding $\cos\theta$ in Eq. (C١).

C.I ***<u>The medium modified</u>*** <u>**NN**-*elastic angular distribution*</u>

The integral form for the $NN-\ NN$ process in Eq. (٢,٢٣) can be rewritten as

$$\frac{dc_{\ NN\rightarrow NN}}{dt}=D_{١}^{NN}+D_{٢}^{NN}+D_{٣}^{NN}+D_{٤}^{NN}, \tag{C٢}$$

where

$$D_{١}^{NN}=\frac{\Lambda_\sigma^{٨}}{٢}\left[\frac{a_{١}^{١}}{\left(\Lambda_\sigma^{٢}-t\right)^{٤}}+\frac{a_{٢}^{١}}{\left(\Lambda_\sigma^{٢}-t\right)^{٣}}+\frac{a_{٣}^{١}}{\left(\Lambda_\sigma^{٢}-t\right)^{٢}}+\frac{a_{٤}^{١}}{\left(\Lambda_\sigma^{٢}-t\right)}+\frac{a_{٥}^{١}}{\left(t-m_\sigma^{٢}\right)^{٢}}+\frac{a_{٦}^{١}}{\left(t-m_\sigma^{٢}\right)}\right], \tag{C٣}$$

$$D_{٢}^{NN}=\Lambda_\omega^{٨}\left[\frac{a_{١}^{٢}}{\left(\Lambda_\omega^{٢}-t\right)^{٤}}+\frac{a_{٢}^{٢}}{\left(\Lambda_\omega^{٢}-t\right)^{٣}}+\frac{a_{٣}^{٢}}{\left(\Lambda_\omega^{٢}-t\right)^{٢}}+\frac{a_{٤}^{٢}}{\left(\Lambda_\omega^{٢}-t\right)}+\frac{a_{٥}^{٢}}{\left(t-m_\omega^{٢}\right)^{٢}}+\frac{a_{٦}^{٢}}{\left(t-m_\omega^{٢}\right)}\right], \tag{C٤}$$

$$D_{٣}^{NN}=٢٤\Lambda_\pi^{٨}\left[\frac{a_{١}^{٣}}{\left(\Lambda_\pi^{٢}-t\right)^{٤}}+\frac{a_{٢}^{٣}}{\left(\Lambda_\pi^{٢}-t\right)^{٣}}+\frac{a_{٣}^{٣}}{\left(\Lambda_\pi^{٢}-t\right)^{٢}}+\frac{a_{٤}^{٣}}{\left(\Lambda_\pi^{٢}-t\right)}+\frac{a_{٥}^{٣}}{\left(t-m_\pi^{٢}\right)^{٢}}+\frac{a_{٦}^{٣}}{\left(t-m_\pi^{٢}\right)}\right], \tag{C٥}$$

$$D_{٤}^{NN} = -٤\Lambda_{\omega}^{٤}\Lambda_{\omega}^{٤}\left[\frac{a_{١}^{٤}}{\left(\Lambda_{\sigma}^{٢}-t\right)^{٢}} + \frac{a_{٢}^{٤}}{\left(\Lambda_{\sigma}^{٢}-t\right)} + \frac{a_{٣}^{٤}}{\left(\Lambda_{\omega}^{٢}-t\right)^{٢}} + \frac{a_{٤}^{٤}}{\left(\Lambda_{\omega}^{٢}-t\right)} + \frac{a_{٥}^{٤}}{\left(t-m_{\sigma}^{٢}\right)} + \frac{a_{٦}^{٤}}{\left(t-m_{\omega}^{٢}\right)}\right],$$

(C٦)

where a_i^j is determined by solving a set of ٦ linear equations.

The coefficients $a_i^{١}$ of the $D_{١}^{NN}$ term is determined from:

$$a_{١}^{١} m_c^{٤} + a_{٢}^{١} m_c^{٤} \Lambda_c^{٢} + a_{٣}^{١} m_c^{٤} \Lambda_c^{٤} + a_{٤}^{١} m_c^{٤} \Lambda_c^{٦} + a_{٥}^{١} \Lambda_c^{٨} - a_{٦}^{١} m_c^{٨} \Lambda_c^{٢} = ١٦ m^{*٤}, \quad \text{(C٧)}$$

$$-٢a_{١}^{١} m_c^{٢} - a_{٢}^{١}\left(٢m_c^{٢}\Lambda_c^{٢} + m_c^{٤}\right) - a_{٣}^{١}\left(٢m_c^{٢}\Lambda_c^{٤} + ٢m_c^{٤}\Lambda_c^{٢}\right) - a_{٤}^{١}\left(٢m_c^{٢}\Lambda_c^{٦}\right.$$
$$\left. + ٣m_c^{٤}\Lambda_c^{٤}\right) - ٤a_{٥}^{١}\Lambda_c^{٦} + a_{٦}^{١}\left(\Lambda^{٨} + ٤m_c^{٢}\Lambda_c^{٦}\right) = -٨m^{*٢},$$

(C٨)

$$a_{١}^{٢} + a_{٢}^{١}\left(\Lambda_c^{٢} + ٢m_c^{٢}\right) + a_{٣}^{١}\left(\Lambda_c^{٤} + ٤m_c^{٢}\Lambda_c^{٢} + m_c^{٤}\right) + a_{٤}^{١}\left(\Lambda_c^{٦} + ٦m_c^{٢}\Lambda_c^{٤}\right.$$
$$\left. + ٣m_c^{٤}\Lambda_c^{٢}\right) + ٦a_{٥}^{١}\Lambda_c^{٤} - a_{٦}^{١}\left(٤\Lambda_c^{٦} + ٦m_c^{٢}\Lambda_c^{٤}\right) = ١, \quad \text{(C٩)}$$

$$-a_{٢}^{١} - a_{٣}^{١}\left(٢\Lambda_c^{٢} + ٢m_c^{٢}\right) - a_{٤}^{١}\left(٣\Lambda_c^{٤} + ٦m_c^{٢}\Lambda_c^{٢} + m_c^{٤}\right) - ٤a_{٥}^{١}\Lambda_c^{٢}$$
$$+ a_{٦}^{١}\left(٦\Lambda_c^{٤} + ٤m_c^{٢}\Lambda_c^{٢}\right) = ٠, \quad \text{(C١٠)}$$

$$a_{٣}^{١} + a_{٤}^{١}\left(٣\Lambda_c^{٢} + ٢m_c^{٢}\right) + a_{٥}^{١} - a_{٦}^{١}\left(٤\Lambda_c^{٢} + m_c^{٢}\right) = ٠, \quad \text{(C١١)}$$

$$-a_{٤}^{١} + a_{٦}^{١} = ٠. \quad \text{(C١٢)}$$

For the $D_{٢}^{NN}$ term, the corresponding $a_i^{٢}$ coefficients are determined from

$$a_{١}^{٢} m_a^{٤} + a_{٢}^{٢} m_a^{٤} \Lambda_a^{٢} + a_{٣}^{٢} m_a^{٤} \Lambda_a^{٤} + a_{٤}^{٢} m_a^{٤} \Lambda_a^{٦} + a_{٥}^{٢} \Lambda_a^{٨} - a_{٦}^{٢} m_a^{٢} \Lambda_a^{٨} = ٢s^{٢} - ٨m^{*٢}s - ٨m^{*٤}, \quad (C١٣)$$

$$-٢a_{١}^{٢} m_a^{٢} - a_{٢}^{٢}\left(٢m_a^{٢}\Lambda_a^{٢} + m_a^{٤}\right) - a_{٣}^{٢}\left(٢m_a^{٢}\Lambda_a^{٤} + ٢m_a^{٤}\Lambda_a^{٢}\right) - a_{٤}^{٢}\left(٢m_a^{٢}\Lambda_a^{٦} + ٣m_a^{٤}\Lambda_a^{٤}\right) - ٤a_{٥}^{٢}\Lambda_a^{٦} + a_{٦}^{٢}\left(\Lambda_a^{٨} + ٤m_a^{٢}\Lambda_a^{٦}\right) = ٢s, \quad (C١٤)$$

$$a_{١}^{٢} + a_{٢}^{٢}\left(\Lambda_a^{٢} + ٢m_a^{٢}\right) + a_{٣}^{٢}\left(\Lambda_a^{٤} + ٤m_a^{٢}\Lambda_a^{٢} + m_a^{٤}\right) + a_{٤}^{٢}\left(\Lambda_a^{٦} + ٦m_a^{٢}\Lambda_a^{٤} + ٣m_a^{٤}\Lambda_a^{٢}\right) + ٦a_{٥}^{٢}\Lambda_a^{٤} - a_{٦}^{٢}\left(٤\Lambda_a^{٦} + ٦m_a^{٤}\Lambda_a^{٢}\right) = ١, \quad (C١٥)$$

$$-a_{٢}^{٢} - a_{٣}^{٢}\left(٢\Lambda_a^{٢} + ٢m_a^{٢}\right) - a_{٤}^{٢}\left(٣\Lambda_a^{٤} + ٦m_a^{٢}\Lambda_a^{٢} + m_a^{٤}\right) - ٤a_{٥}^{٢}\Lambda_a^{٤} + a_{٦}^{٢}\left(٦\Lambda_a^{٤} + ٤m_a^{٢}\Lambda_a^{٢}\right) = ٠, \quad (C١٦)$$

$$a_{٣}^{٢} + a_{٤}^{٢}\left(٣\Lambda_a^{٢} + ٢m_a^{٢}\right) + a_{٥}^{٢} - a_{٦}^{٢}\left(٤\Lambda_a^{٢} + m_a^{٢}\right) = ٠, \quad (C١٧)$$

$$-a_{٤}^{٢} + a_{٦}^{٢} = ٠. \quad (C١٨)$$

As for the $D_{٣}^{NN}$ term, the $a_i^{٣}$ are given by

$$a_{١}^{٣} m_\pi^{٤} + a_{٢}^{٣} m_\pi^{٤}\Lambda_\pi^{٢} + a_{٣}^{٣} m_\pi^{٤}\Lambda_\pi^{٤} + a_{٤}^{٣} m_\pi^{٤}\Lambda_\pi^{٦} + a_{٥}^{٣}\Lambda_\pi^{٨} - a_{٦}^{٣} m_\pi^{٢}\Lambda_\pi^{٨} = ٠, \quad (C١٩)$$

$$-٢a_١^٣ m_\pi^٢ - a_٢^٣\left(٢m_\pi^٢\Lambda_\pi^٢ + m_\pi^٤\right) - a_٣^٣\left(٢m_\pi^٢\Lambda_\pi^٤ + ٢m_\pi^٤\Lambda_\pi^٢\right) - a_٤^٣\left(٢m_\pi^٢\Lambda_\pi^٦ + ٣m_\pi^٤\Lambda_\pi^٤\right) - ٤a_٥^٣\Lambda_\pi^٦ + a_٦^٣\left(\Lambda_\pi^٨ + ٤m_\pi^٢\Lambda_\pi^٦\right) = ٠, \qquad (C٢٠)$$

$$a_١^٣ + a_٢^٣\left(\Lambda_\pi^٢ + ٢m_\pi^٢\right) + a_٣^٣\left(\Lambda_\pi^٤ + ٤m_\pi^٢\Lambda_\pi^٢ + m_\pi^٤\right) + a_٤^٣\left(\Lambda_\pi^٦ + ٦m_\pi^٢\Lambda_\pi^٤ + ٣m_\pi^٤\Lambda_\pi^٢\right) + ٦a_٥^٣\Lambda_\pi^٤ - a_٦^٣\left(٤\Lambda_\pi^٦ + ٦m_\pi^٢\Lambda_\pi^٤\right) = m^{*٤}, \qquad (C٢١)$$

$$-a_٢^٣ - a_٣^٣\left(٢\Lambda_\pi^٢ + ٢m_\pi^٢\right) - a_٤^٣\left(٣\Lambda_\pi^٤ + ٦m_\pi^٢\Lambda_\pi^٢ + m_\pi^٤\right) - ٤a_٥^٣\Lambda_\pi^٢ + a_٦^٣\left(٦\Lambda_\pi^٤ + ٤m_\pi^٢\Lambda_\pi^٢\right) = ٠, \qquad (C٢٢)$$

$$a_٣^٣ + a_٤^٣\left(٣\Lambda_\pi^٢ + ٢m_\pi^٢\right) + a_٥^٣ - a_٦^٣\left(٤\Lambda_\pi^٢ + m_\pi^٢\right) = ٠, \qquad (C٢٣)$$

$$-a_٤^٣ + a_٦^٣ = ٠. \qquad (C٢٤)$$

Finally for the $D_٤^{NN}$ term, the $a_i^٤$ are found by solving

$$a_١^٤ m_a^٢ m_c^٢ \Lambda_a^٤ + a_٢^٤ m_c^٢ m_a^٢ \Lambda_c^٢ \Lambda_a^٤ + a_٣^٤ m_c^٢ m_a^٢ \Lambda_c^٤ + a_٤^٤ m_c^٢ m_a^٢ \Lambda_c^٤ \Lambda_a^٢ - a_٥^٤ m_a^٢ \Lambda_c^٤ \Lambda_a^٤ - a_٦^٤ m_c^٢ \Lambda_c^٤ \Lambda_a^٤ = ٢sm^{*٢} - ٤m^{*٤}, \qquad (C٢٥)$$

$$-a_١^٤\left(\Lambda_a^٤\left(m_c^٢ + m_a^٢\right) + ٢m_c^٢ m_a^٢\Lambda_a^٢\right) - a_٢^٤\left(\Lambda_c^٢\Lambda_a^٤\left(m_c^٢ + m_a^٢\right) - m_c^٢ m_a^٢\Lambda_a^٤ + ٢m_c^٢ m_a^٢\Lambda_c^٢\Lambda_a^٢\right) - a_٣^٤\left(\Lambda_c^٤\left(m_c^٢ + m_a^٢\right) + ٢m_c^٢ m_a^٢\Lambda_c^٢\right) - a_٤^٤\left(\Lambda_c^٤\Lambda_a^٢ m_c^٢ + \Lambda_c^٤\Lambda_a^٢ m_a^٢ + m_c^٢ m_a^٢\Lambda_c^٤ + ٢m_c^٢ m_a^٢\Lambda_c^٢\Lambda_a^٢\right) + a_٥^٤\left(\Lambda_c^٤\Lambda_a^٤ + ٢m_a^٢\Lambda_c^٤\Lambda_a^٢ + ٢m_a^٢\Lambda_c^٢\Lambda_a^٤\right) + a_٦^٤\left(\Lambda_c^٤\Lambda_a^٤ + \left(٢m_c^٢\left(\Lambda_c^٤\Lambda_a^٢ + \Lambda_c^٢\Lambda_a^٤\right)\right)\right) = m^{*٢}, \qquad (C٢٦)$$

$$
\begin{aligned}
& a_{١}^{٤}\left(\Lambda_{a}^{٤}+m_{c}^{٢}m_{a}^{٢}+٢\Lambda_{a}^{٢}\left(m_{c}^{٢}+m_{a}^{٢}\right)\right)+a_{٢}^{٤}\left(\Lambda_{a}^{٤}\left(m_{c}^{٢}+m_{a}^{٢}\right)+٢m_{c}^{٢}m_{a}^{٢}\Lambda_{a}^{٢}+\Lambda_{c}^{٢}\Lambda_{a}^{٤}\right.\\
& \left.+m_{c}^{٢}m_{a}^{٢}\Lambda_{c}^{٢}+٢\Lambda_{c}^{٢}\Lambda_{a}^{٢}\left(m_{c}^{٢}+m_{a}^{٢}\right)\right)+a_{٣}^{٤}\left(\Lambda_{c}^{٤}+m_{c}^{٢}m_{a}^{٢}+٢\Lambda_{c}^{٢}m_{c}^{٢}+٢\Lambda_{c}^{٢}m_{a}^{٢}\right)\\
& +a_{٤}^{٤}\left(\Lambda_{c}^{٤}\left(m_{c}^{٢}+m_{a}^{٢}\right)+٢m_{c}^{٢}m_{a}^{٢}\Lambda_{c}^{٢}+\Lambda_{c}^{٤}\Lambda_{a}^{٢}+m_{c}^{٢}m_{a}^{٢}\Lambda_{a}^{٢}+٢\Lambda_{c}^{٢}\Lambda_{a}^{٢}\left(m_{c}^{٢}+m_{a}^{٢}\right)\right)\\
& -a_{٥}^{٤}\left(m_{a}^{٢}\left(\Lambda_{c}^{٤}+\Lambda_{a}^{٤}\right)+٤m_{a}^{٢}\Lambda_{c}^{٢}\Lambda_{a}^{٢}+٢\Lambda_{c}^{٤}\Lambda_{a}^{٢}+٢\Lambda_{c}^{٢}\Lambda_{a}^{٤}\right)-a_{٦}^{٤}\left(m_{c}^{٢}\left(\Lambda_{c}^{٤}+\Lambda_{a}^{٤}\right)\right.\\
& \left.+٤m_{c}^{٢}\Lambda_{c}^{٢}\Lambda_{a}^{٢}+٢\Lambda_{c}^{٤}\Lambda_{a}^{٢}+٢\Lambda_{c}^{٢}\Lambda_{a}^{٤}\right)=٠, \qquad (C٢٧)
\end{aligned}
$$

$$
\begin{aligned}
& -a_{١}^{٤}\left(m_{c}^{٢}+m_{a}^{٢}+٢\Lambda_{a}^{٢}\right)-a_{٢}^{٤}\left(\Lambda_{a}^{٤}+m_{c}^{٢}m_{a}^{٢}+٢\Lambda_{a}^{٢}\left(m_{c}^{٢}+m_{a}^{٢}\right)+٢\Lambda_{c}^{٢}\Lambda_{a}^{٢}\right.\\
& \left.+\Lambda_{c}^{٢}\left(m_{c}^{٢}+m_{a}^{٢}\right)\right)-a_{٣}^{٤}\left(m_{c}^{٢}+m_{a}^{٢}+٢\Lambda_{c}^{٢}\right)-a_{٤}^{٤}\left(\Lambda_{c}^{٤}+m_{c}^{٢}m_{a}^{٢}+٢\Lambda_{c}^{٢}\Lambda_{a}^{٢}\right.\\
& \left.+٢\Lambda_{c}^{٢}\left(m_{c}^{٢}+m_{a}^{٢}\right)+\Lambda_{a}^{٢}\left(m_{c}^{٢}+m_{a}^{٢}\right)\right)+a_{٥}^{٤}\left(٤\Lambda_{c}^{٢}\Lambda_{a}^{٢}+٢m_{a}^{٢}\Lambda_{c}^{٢}+٢m_{a}^{٢}\Lambda_{a}^{٢}\right.\\
& \left.+\Lambda_{c}^{٤}+\Lambda_{a}^{٤}\right)+a_{٦}^{٤}\left(\Lambda_{c}^{٤}+\Lambda_{a}^{٤}+٤\Lambda_{c}^{٢}\Lambda_{a}^{٢}+٢m_{c}^{٢}\left(\Lambda_{c}^{٢}+\Lambda_{a}^{٢}\right)\right)=٠, \qquad (C٢٨)
\end{aligned}
$$

$$
\begin{aligned}
& a_{١}^{٤}+a_{٢}^{٤}\left(٢\Lambda_{a}^{٢}+\Lambda_{c}^{٢}+m_{a}^{٢}+m_{c}^{٢}\right)+a_{٣}^{٤}+a_{٤}^{٤}\left(٢\Lambda_{c}^{٢}+\Lambda_{a}^{٢}+m_{a}^{٢}+m_{c}^{٢}\right)\\
& -a_{٥}^{٤}\left(٢\left(\Lambda_{a}^{٢}+\Lambda_{c}^{٢}\right)+m_{a}^{٢}\right)-a_{٦}^{٤}\left(٢\left(\Lambda_{a}^{٢}+\Lambda_{c}^{٢}\right)+m_{c}^{٢}\right)=٠, \qquad (C٢٩)\\
& -a_{٢}^{٤}-a_{٤}^{٤}+a_{٥}^{٤}+a_{٦}^{٤}=٠. \qquad (C٣٠)
\end{aligned}
$$

The sets of equations, C٧-C١٢, C١٣-C١٨, C١٩-C٢٤, and C٢٥-C٣٠, are then inserted into the Gaussian Elimination subprogram (C.III) to find the values of these coefficients.

Substituting Eqs. C٢ and C٣- C٦ into Eq. C١, we get

$$
\xi=\frac{\sum_{i=١}^{٤} D_{ii}^{NN}}{\sum_{i=١}^{٤} A_{ii}^{NN}}, \qquad (C٣١)
$$

where

$$D_{١١}^{NN}=-\frac{\Lambda_{\sigma}^{٨}}{٢}\left[\frac{a_{١}^{١}}{٣\Lambda_{\sigma}^{٦}}\frac{١}{\left(١+t/\Lambda_{\sigma}^{٢}\right)^{٣}}+\frac{a_{٢}^{١}}{٢\Lambda_{\sigma}^{٤}}\frac{١}{\left(١+t/\Lambda_{\sigma}^{٢}\right)^{٢}}+\frac{a_{٣}^{١}}{\Lambda_{\sigma}^{٢}}\frac{١}{\left(١+t/\Lambda_{\sigma}^{٢}\right)}\right.$$

$$-a_{٤}^{١}\ln\left(١+t/\Lambda_{\sigma}^{٢}\right)+\frac{a_{٥}^{١}}{m_{c}^{٢}}\frac{١}{\left(١+t/m_{c}^{٢}\right)}+a_{٦}^{١}\ln\left(١+t/m_{\sigma}^{٢}\right)-\frac{a_{١}^{١}}{٣\Lambda_{c}^{٦}}$$

$$\left.-\frac{a_{٢}^{١}}{٢\Lambda_{\sigma}^{٤}}-\frac{a_{٣}^{١}}{\Lambda_{\sigma}^{٢}}-\frac{a_{٥}^{١}}{m_{\sigma}^{٢}}\right], \qquad (C٣٢)$$

$$D_{٢٢}^{NN}=-\Lambda_{\omega}^{٨}\left[\frac{a_{١}^{١}}{٣\Lambda_{\omega}^{٦}}\frac{١}{\left(١+t/\Lambda_{\omega}^{٢}\right)^{٣}}+\frac{a_{٢}^{١}}{٢\Lambda_{\omega}^{٤}}\frac{١}{\left(١+t/\Lambda_{\omega}^{٢}\right)^{٢}}+\frac{a_{٣}^{١}}{\Lambda_{\omega}^{٢}}\frac{١}{\left(١+t/\Lambda_{\omega}^{٢}\right)}\right.$$

$$-a_{٤}^{١}\ln\left(١+t/\Lambda_{\omega}^{٢}\right)+\frac{a_{٥}^{١}}{m_{a}^{٢}}\frac{١}{\left(١+t/m_{a}^{٢}\right)}+a_{٦}^{١}\ln\left(١+t/m_{\omega}^{٢}\right)-\frac{a_{١}^{١}}{٣\Lambda_{a}^{٦}}$$

$$\left.-\frac{a_{٢}^{١}}{٢\Lambda_{\omega}^{٤}}-\frac{a_{٣}^{١}}{\Lambda_{\omega}^{٢}}-\frac{a_{٥}^{١}}{m_{\omega}^{٢}}\right], \qquad (C٣٣)$$

$$D_{٣٣}^{NN}=-٢٤\Lambda_{\pi}^{٨}\left[\frac{a_{١}^{١}}{٣\Lambda_{\pi}^{٦}}\frac{١}{\left(١+t/\Lambda_{\pi}^{٢}\right)^{٣}}+\frac{a_{٢}^{١}}{٢\Lambda_{\pi}^{٤}}\frac{١}{\left(١+t/\Lambda_{\pi}^{٢}\right)^{٢}}+\frac{a_{٣}^{١}}{\Lambda_{\pi}^{٢}}\frac{١}{\left(١+t/\Lambda_{\pi}^{٢}\right)}\right.$$

$$-a_{٤}^{١}\ln\left(١+t/\Lambda_{\pi}^{٢}\right)+\frac{a_{٥}^{١}}{m_{\pi}^{٢}}\frac{١}{\left(١+t/m_{\pi}^{٢}\right)}+a_{٦}^{١}\ln\left(١+t/m_{\pi}^{٢}\right)-\frac{a_{١}^{١}}{٣\Lambda_{\pi}^{٦}}$$

$$\left.-\frac{a_{٢}^{١}}{٢\Lambda_{\pi}^{٤}}-\frac{a_{٣}^{١}}{\Lambda_{\pi}^{٢}}-\frac{a_{٥}^{١}}{m_{\pi}^{٢}}\right], \qquad (C٣٤)$$

$$D_{٤٤}^{NN}=٤\Lambda_{\sigma}^{٤}\Lambda_{\omega}^{٤}\left[\frac{a_{١}^{٤}}{\Lambda_{\sigma}^{٢}}\frac{١}{\left(١+t/\Lambda_{\sigma}^{٢}\right)}-a_{٢}^{٤}\ln\left(١+t/\Lambda_{\sigma}^{٢}\right)+\frac{a_{٣}^{٤}}{\Lambda_{\omega}^{٢}}\frac{١}{\left(١+t/\Lambda_{\omega}^{٢}\right)}\right.$$

$$-a_{٤}^{٤}\ln\left(١+t/\Lambda_{\omega}^{٢}\right)+a_{٥}^{٤}\ln\left(١+t/m_{\sigma}^{٢}\right)+a_{٦}^{٤}\ln\left(١+t/m_{\omega}^{٢}\right)-\frac{a_{١}^{٤}}{\Lambda_{c}^{٢}}-\frac{a_{٣}^{٤}}{\Lambda_{a}^{٢}},\quad (C٣٥)$$

$$A_{١١}^{NN}=-\frac{\Lambda_{\sigma}^{٨}}{٢}\left[\frac{a_{١}^{١}}{٣\Lambda_{\sigma}^{٦}}\frac{١}{\left(١+t_{\max}/\Lambda_{\sigma}^{٢}\right)^{٣}}+\frac{a_{٢}^{١}}{٢\Lambda_{\sigma}^{٤}}\frac{١}{\left(١+t_{\max}/\Lambda_{\sigma}^{٢}\right)^{٢}}+\frac{a_{٣}^{١}}{\Lambda_{\sigma}^{٢}}\frac{١}{\left(١+t_{\max}/\Lambda_{\sigma}^{٢}\right)}\right.$$
$$-a_{٤}^{١}\ln\left(١+t_{\max}/\Lambda_{\sigma}^{٢}\right)+\frac{a_{٥}^{١}}{m_{c}^{٢}}\frac{١}{\left(١+t_{\max}/m_{c}^{٢}\right)}+a_{٦}^{١}\ln\left(١+t_{\max}/m_{\sigma}^{٢}\right)-\frac{a_{١}^{١}}{٣\Lambda_{c}^{٦}}$$
$$\left.-\frac{a_{٢}^{١}}{٢\Lambda_{\sigma}^{٤}}-\frac{a_{٣}^{١}}{\Lambda_{\sigma}^{٢}}-\frac{a_{٥}^{١}}{m_{\sigma}^{٢}}\right],\quad (C٣٦)$$

$$A_{٢٢}^{NN}=-\Lambda_{\omega}^{٨}\left[\frac{a_{١}^{١}}{٣\Lambda_{\omega}^{٦}}\frac{١}{\left(١+t_{\max}/\Lambda_{\omega}^{٢}\right)^{٣}}+\frac{a_{٢}^{١}}{٢\Lambda_{\omega}^{٤}}\frac{١}{\left(١+t_{\max}/\Lambda_{\omega}^{٢}\right)^{٢}}+\frac{a_{٣}^{١}}{\Lambda_{\omega}^{٢}}\frac{١}{\left(١+t_{\max}/\Lambda_{\omega}^{٢}\right)}\right.$$
$$-a_{٤}^{١}\ln\left(١+t_{\max}/\Lambda_{\omega}^{٢}\right)+\frac{a_{٥}^{١}}{m_{a}^{٢}}\frac{١}{\left(١+t_{\max}/m_{a}^{٢}\right)}+a_{٦}^{١}\ln\left(١+t_{\max}/m_{\omega}^{٢}\right)-\frac{a_{١}^{١}}{٣\Lambda_{a}^{٦}}$$
$$\left.-\frac{a_{٢}^{١}}{٢\Lambda_{\omega}^{٤}}-\frac{a_{٣}^{١}}{\Lambda_{\omega}^{٢}}-\frac{a_{٥}^{١}}{m_{\omega}^{٢}}\right],\quad (C٣٧)$$

$$A_{٣٣}^{NN}=-٢٤\Lambda_{\pi}^{٨}\left[\frac{a_{١}^{١}}{٣\Lambda_{\pi}^{٦}}\frac{١}{\left(١+t_{\max}/\Lambda_{\pi}^{٢}\right)^{٣}}+\frac{a_{٢}^{١}}{٢\Lambda_{\pi}^{٤}}{}^{٦}\frac{١}{\left(١+t_{\max}/\Lambda_{\pi}^{٢}\right)^{٢}}+\frac{a_{٣}^{١}}{\Lambda_{\pi}^{٢}}\frac{١}{\left(١+t_{\max}/\Lambda_{\pi}^{٢}\right)}\right.$$
$$-a_{٤}^{١}\ln\left(١+t_{\max}/\Lambda_{\pi}^{٢}\right)+\frac{a_{٥}^{١}}{m_{\pi}^{٢}}\frac{١}{\left(١+t_{\max}/m_{\pi}^{٢}\right)}+a_{٦}^{١}\ln\left(١+t_{\max}/m_{\pi}^{٢}\right)-\frac{a_{١}^{١}}{٣\Lambda_{\pi}^{٦}}$$
$$\left.-\frac{a_{٢}^{١}}{٢\Lambda_{\pi}^{٤}}-\frac{a_{٣}^{١}}{\Lambda_{\pi}^{٢}}-\frac{a_{٥}^{١}}{m_{\pi}^{٢}}\right],\quad (C٣٨)$$

$$A_{٤٤}^{NN}=٤\Lambda_{\sigma}^{٤}\Lambda_{\omega}^{٤}\left[\frac{a_{١}^{٤}}{\Lambda_{\sigma}^{٢}}\frac{١}{\left(١+t_{\max}/\Lambda_{\sigma}^{٢}\right)}-a_{٢}^{٤}\ln\left(١+t_{\max}/\Lambda_{\sigma}^{٢}\right)+\frac{a_{٣}^{٤}}{\Lambda_{\omega}^{٢}}\frac{١}{\left(١+t_{\max}/\Lambda_{\omega}^{٢}\right)}\right.$$

$$-a_{٤}^{٤}\ln\left(١+t_{\max}/\Lambda_{\omega}^{٢}\right)+a_{٥}^{٤}\ln\left(١+t_{\max}/m_{\sigma}^{٢}\right)+a_{٦}^{٤}\ln\left(١+t_{\max}/m_{\omega}^{٢}\right)-\frac{a_{١}^{٤}}{\Lambda_{c}^{٢}}-\frac{a_{٣}^{٤}}{\Lambda_{a}^{٢}},$$

(C٣٩)

where
$$t=\frac{١}{٢}\left(s-٤m^{*٢}\right)\left(\cos\theta-١\right), \qquad (A٢٩)$$

$$t_{\max}=\left|s-٤m^{*٢}\right|. \qquad (C٤٠)$$

C.II ***The medium modified* NN-*inelastic angular distribution***

Similar to the $NN-NN$ process, we can rewrite Eq. C١ for the $NN-N\Delta$ process as:

$$\xi=\frac{D^{N\Delta}+\sum_{i=١}^{٦}E_{i}^{N\Delta}}{\sum_{i=٠}^{٦}A_{i}^{N\Delta}}. \qquad (C٤١)$$

Where the direct term $D^{N\Delta}$ is given by

$$D^{N\Delta} = -\frac{m^{*٢}}{٦ m_{\Delta}^{*٢}}\left\{\frac{a_{١}^{D}}{٣\Lambda_{\pi}^{٦}}\frac{١}{\left(١+t/\Lambda_{\pi}^{٢}\right)^{٣}}+\frac{a_{٢}^{D}}{٢\Lambda_{\pi}^{٤}}\frac{١}{\left(١+t/\Lambda_{\pi}^{٢}\right)^{٢}}\right.$$

$$+\frac{a_{٣}^{D}}{\Lambda_{\pi}^{٢}}\frac{١}{\left(١+t/\Lambda_{\pi}^{٢}\right)}+\frac{a_{٥}^{D}}{m_{\pi}^{٢}}\frac{١}{\left(١+t/m_{\pi}^{٢}\right)}+a_{٤}^{D}\ln\left(\frac{١+t/m_{\pi}^{٢}}{١+t/\Lambda_{\pi}^{٢}}\right)$$

$$\left.-\left(\frac{a_{١}^{D}}{٣\Lambda_{\pi}^{٦}}+\frac{a_{٢}^{D}}{٢\Lambda_{\pi}^{٤}}+\frac{a_{٣}^{D}}{\Lambda_{\pi}^{٢}}+\frac{a_{٥}^{D}}{m_{\pi}^{٢}}\right)\right\}. \tag{C٤٢}$$

The exchange terms are given by

$$E_{١}^{N\Delta} = -\frac{m^{*٢}}{١٢}\left\{\frac{a_{١}^{E١}}{\Lambda_{\pi}^{٢}}\left(\frac{١}{١+t/\Lambda_{\pi}^{٢}}-١\right)-\frac{a_{٣}^{E١}}{\Lambda_{\nu}^{٢}}\left(\frac{١}{١-t/\Lambda_{\nu}^{٢}}-١\right)\right.$$

$$-a_{٢}^{E١}\ln\left(١+t/\Lambda_{\pi}^{٢}\right)+a_{٤}^{E١}\ln\left(١-t/\Lambda_{\nu}^{٢}\right)+a_{٥}^{E١}\ln\left(١+t/m_{\pi}^{٢}\right)$$

$$\left.-a_{٦}^{E١}\ln(١+t/a)\right\}, \tag{C٤٣}$$

$$E_{٢}^{N\Delta} = -\frac{m^{*٣}}{١٢}m_{\Delta}^{*}\left\{\frac{a_{١}^{E١}}{\Lambda_{\pi}^{٢}}\left(\frac{١}{١+t/\Lambda_{\pi}^{٢}}-١\right)-\frac{a_{٣}^{E١}}{\Lambda_{\nu}^{٢}}\left(\frac{١}{١-t/\Lambda_{\nu}^{٢}}-١\right)\right.$$

$$-a_{٢}^{E٢}\ln\left(١+t/\Lambda_{\pi}^{٢}\right)+a_{٤}^{E٢}\ln\left(١-t/\Lambda_{\nu}^{٢}\right)+a_{٥}^{E٢}\ln\left(١+t/m_{\pi}^{٢}\right)$$

$$\left.-a_{٦}^{E٢}\ln(١+t/a)\right\}, \tag{C٤٤}$$

$$E_{٣}^{N\Delta} = -\frac{m^{*٣}}{١٢ m_{\Delta}^{*}}\left\{\frac{a_{١}^{E٣}}{\Lambda_{\pi}^{٢}}\left(\frac{١}{١+t/\Lambda_{\pi}^{٢}}-١\right)-\frac{a_{٣}^{E٣}}{\Lambda_{\nu}^{٢}}\left(\frac{١}{١-t/\Lambda_{\nu}^{٢}}-١\right)\right.$$

$$-a_{٢}^{E٣}\ln\left(١+t/\Lambda_{\pi}^{٢}\right)+a_{٤}^{E٣}\ln\left(١-t/\Lambda_{\nu}^{٢}\right)+a_{٥}^{E٣}\ln\left(١+t/m_{\pi}^{٢}\right)$$

$$\left.-a_{٦}^{E٣}\ln(١+t/a)\right\}, \tag{C٤٥}$$

$$E_{٤}^{N\Delta} = -\frac{m^{*٢}}{١٢ m_{\Delta}^{*٢}}\Bigg\{ \frac{a_{١}^{E٤}}{\Lambda_{\pi}^{٢}}\left(\frac{١}{١+t/\Lambda_{\pi}^{٢}} - ١\right) - \frac{a_{٣}^{E٤}}{\Lambda_{\nu}^{٢}}\left(\frac{١}{١-t/\Lambda_{\nu}^{٢}} - ١\right)$$

$$- a_{٢}^{E٤}\ln\left(١+t/\Lambda_{\pi}^{٢}\right) + a_{٤}^{E٤}\ln\left(١-t/\Lambda_{\nu}^{٢}\right) + a_{٥}^{E٤}\ln\left(١+t/m_{\pi}^{٢}\right)$$

$$- a_{٦}^{E٤}\ln\left(١+t/a\right)\Bigg\}, \qquad (C٤٦)$$

$$E_{٥}^{N\Delta} = E_{٦}^{N\Delta} = E_{٤}^{N\Delta}. \qquad (C٤٧)$$

The factors $A_{٠}^{N\Delta}$, $A_{١}^{N\Delta}$, $A_{٢}^{N\Delta}$, $A_{٣}^{N\Delta}$, $A_{٤}^{N\Delta}$, $A_{٥}^{N\Delta}$, and $A_{٦}^{N\Delta}$ are given by replacing t with $t_{mx} = \frac{١}{٢}\left(٣m^{*٢} + ٢m_{\Delta}^{*٢} - s\right) + ٢|p||p_{٣}|$ in $D^{N\Delta}$, $E_{١}^{N\Delta}$, $E_{٢}^{N\Delta}$, $E_{٣}^{N\Delta}$, $E_{٤}^{N\Delta}$, $E_{٥}^{N\Delta}$, and $E_{٦}^{N\Delta}$, respectively.

Where $a = ٣m^{*٢} - s - m_{\pi}^{٢} + m_{\Delta}^{*٢}$ and $\Lambda_{\nu}^{٢} = \Lambda_{\pi}^{٢} - ٣m^{*٢} - m_{\Delta}^{*٢} + s$.

The coefficients a_{i}^{D} in Eq. (C٤٢) are determined by solving:

$$a_{١}^{D} m_{\pi}^{٤} + a_{٢}^{D} m_{\pi}^{٤}\Lambda_{\pi}^{٢} + a_{٣}^{D} m_{\pi}^{٤}\Lambda_{\pi}^{٤} + a_{٤}^{D} m_{\pi}^{٤}\Lambda_{\pi}^{٦} + a_{٥}^{D}\Lambda_{\pi}^{٨} - a_{٦}^{D} m_{\pi}^{٢}\Lambda_{\pi}^{٨} = ٠, \quad (C٤٨)$$

$$- ٢a_{١}^{D} m_{\pi}^{٢} - a_{٢}^{D}\left(٢m_{\pi}^{٢}\Lambda_{\pi}^{٢} + m_{\pi}^{٤}\right) - a_{٣}^{D}\left(٢m_{\pi}^{٢}\Lambda_{\pi}^{٤} + ٢m_{\pi}^{٤}\Lambda_{\pi}^{٢}\right)$$

$$- a_{٤}^{D}\left(٢m_{\pi}^{٢}\Lambda_{\pi}^{٦} + ٣m_{\pi}^{٤}\Lambda_{\pi}^{٤}\right) - ٤a_{٥}^{D}\Lambda_{\pi}^{٦} + a_{٦}^{D}\left(\Lambda_{\pi}^{٨} + ٤m_{\pi}^{٢}\Lambda_{\pi}^{٦}\right)$$

$$= \left(m_{\Delta}^{*} + m^{*}\right)^{٤}\left(m_{\Delta}^{*} - m^{*}\right)^{٢}, \qquad (C٤٩)$$

$$a_{١}^{D} + a_{٢}^{D}\left(\Lambda_{\pi}^{٢} + ٢m_{\pi}^{٢}\right) + a_{٣}^{D}\left(\Lambda_{\pi}^{٤} + ٤m_{\pi}^{٢}\Lambda_{\pi}^{٢} + m_{\pi}^{٤}\right) + ٦a_{٥}^{D}\Lambda_{\pi}^{٤}$$

$$+ a_{٤}^{D}\left(\Lambda_{\pi}^{٦} + ٦ m_{\pi}^{٢}\Lambda_{\pi}^{٤} + ٣ m_{\pi}^{٤}\Lambda_{\pi}^{٢}\right) - a_{٦}^{D}\left(٤\Lambda_{\pi}^{٦} + ٦ m_{\pi}^{٢}\Lambda_{\pi}^{٤}\right)$$

$$= -٢\left(m_{\Delta}^{*} + m^{*}\right)^{٢}\left(m_{\Delta}^{*} - m^{*}\right)^{٢} - \left(m_{\Delta}^{*} + m^{*}\right)^{٤}, \qquad (C٥٠)$$

$$- a_{٢}^{D} - ٢a_{٣}^{D}\left(\Lambda_{\pi}^{٢} + m_{\pi}^{٢}\right) - a_{٤}^{D}\left(٣\Lambda_{\pi}^{٤} + ٦ m_{\pi}^{٢}\Lambda_{\pi}^{٢} + m_{\pi}^{٤}\right) - ٤a_{٥}^{D}\Lambda_{\pi}^{٢}$$

$$+ a_{٦}^{D}\left(٦\Lambda_{\pi}^{٤} + ٤ m_{\pi}^{٢}\Lambda_{\pi}^{٢}\right) = ٢\left(m_{\Delta}^{*} + m^{*}\right)^{٢} + \left(m_{\Delta}^{*} - m^{*}\right)^{٢}, \qquad (C٥١)$$

$$a_{٣}^{D} + a_{٤}^{D}\left(٣\Lambda_{\pi}^{٢} + ٢ m_{\pi}^{٢}\right) + a_{٥}^{D} - a_{٦}^{D}\left(٤\Lambda_{\pi}^{٢} + m_{\pi}^{٢}\right) = -١, \qquad (C٥٢)$$

$$+ a_{٤}^{D} - a_{٦}^{D} = ٠. \qquad (C٥٣)$$

For the $a_i^{E١}$ coefficients:

$$a_{١}^{E١} a m_{\pi}^{٢}\Lambda_{\nu}^{٤} + a_{٢}^{E١} a m_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤} + a_{٣}^{E١} a m_{\pi}^{٢}\Lambda_{\pi}^{٤} + a_{٤}^{E١} a m_{\pi}^{٢}\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}$$

$$- a_{٥}^{E١} a\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤} + a_{٦}^{E١} m_{\pi}^{٢}\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤} = -٢ s m^{*٤} + ٣ m^{*٦},$$

(C٥٤)

$$+ a_{١}^{E١}\left(-\Lambda_{\nu}^{٤}\left(a + m_{\pi}^{٢}\right) + ٢ a m_{\pi}^{٢}\Lambda_{\nu}^{٢}\right) + a_{٢}^{E١}\left(-\Lambda_{\nu}^{٤}\Lambda_{\pi}^{٢}\left(a + m_{\pi}^{٢}\right)\right.$$

$$\left. + ٢ a m_{\pi}^{٢}\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢} - a m_{\pi}^{٢}\Lambda_{\nu}^{٤}\right) - a_{٣}^{E١}\left(\Lambda_{\pi}^{٤}\left(a + m_{\pi}^{٢}\right) + ٢ a m_{\pi}^{٢}\Lambda_{\pi}^{٢}\right)$$

$$+ a_{٤}^{E١}\left(-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\left(a + m_{\pi}^{٢}\right) - ٢ a m_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢} + a m_{\pi}^{٢}\Lambda_{\pi}^{٤}\right) + a_{٥}^{E١}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}\right.$$

$$\left. + ٢a\left(\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤} - \Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\right)\right) - a_{٦}^{E١}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤} + ٢ m_{\pi}^{٢}\left(\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤} - \Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\right)\right)$$

$$= ٨ s m^{*٢} - ٦ m^{*٤} - ٢ s^{٢}, \qquad (C٥٥)$$

$$a_{١}^{E١}\left(\Lambda_{\nu}^{٤} - ٢\Lambda_{\nu}^{٢}\left(a + m_{\pi}^{٢}\right) + a m_{\pi}^{٢}\right) + a_{٢}^{E١}\left(\Lambda_{\nu}^{٤}\Lambda_{\pi}^{٢} - ٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\left(a + m_{\pi}^{٢}\right)\right.$$

$$\left. + \Lambda_{\nu}^{٤}\left(a + m_{\pi}^{٢}\right) + a m_{\pi}^{٢}\Lambda_{\pi}^{٢} - ٢ a m_{\pi}^{٢}\Lambda_{\nu}^{٢}\right) + a_{٣}^{E١}\left(\Lambda_{\pi}^{٤} + ٢\Lambda_{\pi}^{٢}\left(a + m_{\pi}^{٢}\right) + a m_{\pi}^{٢}\right)$$

$$+a_{٤}^{E١}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}+٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)-\Lambda_{\pi}^{٤}\left(a+m_{\pi}^{٢}\right)+am_{\pi}^{٢}\Lambda_{\nu}^{٢}-٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\right)$$

$$+a_{٥}^{E١}\left(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-a\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}\right)+٤a\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\right)-a_{٦}^{E١}\left(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}\right.$$

$$\left.-m_{\pi}^{٢}\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}\right)+٤m_{\pi}^{٢}\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\right)=-٣m^{*٢}, \qquad (C٥٦)$$

$$a_{١}^{E١}\left(٢\Lambda_{\nu}^{٢}-a-m_{\pi}^{٢}\right)+a_{٢}^{E١}\left(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٤}-a\Lambda_{\pi}^{٢}+٢\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$

$$\left.-m_{\pi}^{٢}\left(a+\Lambda_{\pi}^{٢}\right)\right)+a_{٣}^{E١}\left(-٢\Lambda_{\pi}^{٢}-a-m_{\pi}^{٢}\right)+a_{٤}^{E١}\left(\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}-\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$

$$\left.+٢\Lambda_{\pi}^{٢}\left(a+m_{\pi}^{٢}\right)+am_{\pi}^{٢}\right)+a_{٥}^{E١}\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}-٤\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+٢a\left(\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٢}\right)\right)$$

$$-a_{٦}^{E١}\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}-٤\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+٢m_{\pi}^{٢}\left(\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٢}\right)\right)=٢,$$

(C٥٧)

$$a_{١}^{E١}+a_{٢}^{E١}\left(\Lambda_{\pi}^{٢}-٢\Lambda_{\nu}^{٢}+a+m_{\pi}^{٢}\right)+a_{٣}^{E١}+a_{٤}^{E١}\left(\Lambda_{\nu}^{٢}-٢\Lambda_{\pi}^{٢}-a-m_{\pi}^{٢}\right)$$

$$+a_{٥}^{E١}\left(٢\left(\Lambda_{\nu}^{٢}-\Lambda_{\pi}^{٢}\right)-a\right)-a_{٦}^{E١}\left(٢\left(\Lambda_{\nu}^{٢}-\Lambda_{\pi}^{٢}\right)-m_{\pi}^{٢}\right)=٠, \qquad (C٥٨)$$

$$-a_{٢}^{E١}+a_{٤}^{E١}+a_{٥}^{E١}-a_{٦}^{E١}=٠. \qquad (C٥٩)$$

For the $a_i^{E٢}$ terms:

$$a_{١}^{E٢}am_{\pi}^{٢}\Lambda_{\nu}^{٤}+a_{٢}^{E٢}am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}+a_{٣}^{E٢}am_{\pi}^{٢}\Lambda_{\pi}^{٤}+a_{٤}^{E٢}am_{\pi}^{٢}\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}$$

$$-a_{٥}^{E٢}a\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}+a_{٦}^{E٢}m_{\pi}^{٢}\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}=-٢sm^{*٢}+٦m^{*٤}, \qquad (C٦٠)$$

$$+a_{١}^{E٢}\left(-\Lambda_{\nu}^{٤}\left(a+m_{\pi}^{٢}\right)+٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\right)+a_{٢}^{E٢}\left(-\Lambda_{\nu}^{٤}\Lambda_{\pi}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$

$$\left.+٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}-am_{\pi}^{٢}\Lambda_{\nu}^{٤}\right)-a_{٣}^{E٢}\left(\Lambda_{\pi}^{٤}\left(a+m_{\pi}^{٢}\right)+٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\right)$$

$$+a_{٤}^{E٢}\left(-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)-٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+am_{\pi}^{٢}\Lambda_{\pi}^{٤}\right)+a_{٥}^{E٢}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}\right.$$

$$+٢a(\Lambda_\pi^{٢}\Lambda_\nu^{٤}-\Lambda_\pi^{٤}\Lambda_\nu^{٢}))-a_{٦}^{E٢}(\Lambda_\pi^{٤}\Lambda_\nu^{٤}+٢m_\pi^{٢}(\Lambda_\pi^{٢}\Lambda_\nu^{٤}-\Lambda_\pi^{٤}\Lambda_\nu^{٢}))$$
$$=٢s-٢m^{*٢}, \qquad (C٦١)$$

$$a_{١}^{E٢}(\Lambda_\nu^{٤}-٢\Lambda_\nu^{٢}(a+m_\pi^{٢})+am_\pi^{٢})+a_{٢}^{E٢}(\Lambda_\nu^{٤}\Lambda_\pi^{٢}-٢\Lambda_\nu^{٢}\Lambda_\pi^{٢}(a+m_\pi^{٢})$$
$$+\Lambda_\nu^{٤}(a+m_\pi^{٢})+am_\pi^{٢}\Lambda_\pi^{٢}-٢am_\pi^{٢}\Lambda_\nu^{٢})+a_{٣}^{E٢}(\Lambda_\pi^{٤}+٢\Lambda_\pi^{٢}(a+m_\pi^{٢})+am_\pi^{٢})$$
$$+a_{٤}^{E٢}(\Lambda_\pi^{٤}\Lambda_\nu^{٢}+٢\Lambda_\pi^{٢}\Lambda_\nu^{٢}(a+m_\pi^{٢})-\Lambda_\pi^{٤}(a+m_\pi^{٢})+am_\pi^{٢}\Lambda_\nu^{٢}-٢am_\pi^{٢}\Lambda_\pi^{٢})$$
$$+a_{٥}^{E٢}(٢\Lambda_\nu^{٢}\Lambda_\pi^{٤}-٢\Lambda_\pi^{٢}\Lambda_\nu^{٤}-a(\Lambda_\pi^{٤}+\Lambda_\nu^{٤})+٤a\Lambda_\nu^{٢}\Lambda_\pi^{٢})-a_{٦}^{E٢}(٢\Lambda_\nu^{٢}\Lambda_\pi^{٤}-٢\Lambda_\pi^{٢}\Lambda_\nu^{٤}$$
$$-m_\pi^{٢}(\Lambda_\pi^{٤}+\Lambda_\nu^{٤})+٤m_\pi^{٢}\Lambda_\nu^{٢}\Lambda_\pi^{٢})=١, \qquad (C٦٢)$$

$$a_{١}^{E٢}(٢\Lambda_\nu^{٢}-a-m_\pi^{٢})+a_{٢}^{E٢}(٢\Lambda_\nu^{٢}\Lambda_\pi^{٢}-\Lambda_\nu^{٤}-a\Lambda_\pi^{٢}+٢\Lambda_\nu^{٢}(a+m_\pi^{٢})$$
$$-m_\pi^{٢}(a+\Lambda_\pi^{٢}))+a_{٣}^{E٢}(-٢\Lambda_\pi^{٢}-a-m_\pi^{٢})+a_{٤}^{E٢}(\Lambda_\pi^{٤}-٢\Lambda_\pi^{٢}\Lambda_\nu^{٢}-\Lambda_\nu^{٢}(a+m_\pi^{٢})$$
$$+٢\Lambda_\pi^{٢}(a+m_\pi^{٢})+am_\pi^{٢})+a_{٥}^{E٢}(\Lambda_\pi^{٤}+\Lambda_\nu^{٤}-٤\Lambda_\pi^{٢}\Lambda_\nu^{٢}+٢a(\Lambda_\pi^{٢}-\Lambda_\nu^{٢}))$$
$$-a_{٦}^{E٢}(\Lambda_\pi^{٤}+\Lambda_\nu^{٤}-٤\Lambda_\pi^{٢}\Lambda_\nu^{٢}+٢m_\pi^{٢}(\Lambda_\pi^{٢}-\Lambda_\nu^{٢}))=٠, \qquad (C٦٣)$$

$$a_{١}^{E٢}+a_{٢}^{E٢}(\Lambda_\pi^{٢}-٢\Lambda_\nu^{٢}+a+m_\pi^{٢})+a_{٣}^{E٢}+a_{٤}^{E٢}(\Lambda_\nu^{٢}-٢\Lambda_\pi^{٢}-a-m_\pi^{٢})$$
$$+a_{٥}^{E٢}(٢(\Lambda_\nu^{٢}-\Lambda_\pi^{٢})-a)-a_{٦}^{E٢}(٢(\Lambda_\nu^{٢}-\Lambda_\pi^{٢})-m_\pi^{٢})=٠, \qquad (C٦٤)$$

$$-a_{٢}^{E٢}+a_{٤}^{E٢}+a_{٥}^{E٢}-a_{٦}^{E٢}=٠. \qquad (C٦٥)$$

For the $a_i^{E٣}$ terms:

$$a_{١}^{E٣}am_{\pi}^{٢}\Lambda_{\nu}^{٤}+a_{٢}^{E٣}am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}+a_{٣}^{E٣}am_{\pi}^{٢}\Lambda_{\pi}^{٤}+a_{٤}^{E٣}am_{\pi}^{٢}\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}$$

$$-a_{٥}^{E٣}a\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}+a_{٦}^{E٣}m_{\pi}^{٢}\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}=sm^{*٤}-٣m^{*٦}, \qquad (C٦٦)$$

$$+a_{١}^{E٣}\left(-\Lambda_{\nu}^{٤}\left(a+m_{\pi}^{٢}\right)+٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\right)+a_{٢}^{E٣}\left(-\Lambda_{\nu}^{٤}\Lambda_{\pi}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$

$$\left.+٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}-am_{\pi}^{٢}\Lambda_{\nu}^{٤}\right)-a_{٣}^{E٣}\left(\Lambda_{\pi}^{٤}\left(a+m_{\pi}^{٢}\right)+٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\right)$$

$$+a_{٤}^{E٣}\left(-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)-٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+am_{\pi}^{٢}\Lambda_{\pi}^{٤}\right)+a_{٥}^{E٣}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}\right.$$

$$\left.+٢a\left(\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\right)\right)-a_{٦}^{E٣}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}+٢m_{\pi}^{٢}\left(\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\right)\right)$$

$$=٢sm^{*٢}+s^{٢}+٣m^{*٤}, \qquad (C٦٧)$$

$$a_{١}^{E٣}\left(\Lambda_{\nu}^{٤}-٢\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)+am_{\pi}^{٢}\right)+a_{٢}^{E٣}\left(\Lambda_{\nu}^{٤}\Lambda_{\pi}^{٢}-٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$

$$\left.+\Lambda_{\nu}^{٤}\left(a+m_{\pi}^{٢}\right)+am_{\pi}^{٢}\Lambda_{\pi}^{٢}-٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\right)+a_{٣}^{E٣}\left(\Lambda_{\pi}^{٤}+٢\Lambda_{\pi}^{٢}\left(a+m_{\pi}^{٢}\right)+am_{\pi}^{٢}\right)$$

$$+a_{٤}^{E٣}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}+٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)-\Lambda_{\pi}^{٤}\left(a+m_{\pi}^{٢}\right)+am_{\pi}^{٢}\Lambda_{\nu}^{٢}-٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\right)$$

$$+a_{٥}^{E٣}\left(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-a\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}\right)+٤a\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\right)-a_{٦}^{E٣}\left(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}\right.$$

$$\left.-m_{\pi}^{٢}\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}\right)+٤m_{\pi}^{٢}\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\right)=s-m^{*٢}, \qquad (C٦٨)$$

$$a_{١}^{E٣}\left(٢\Lambda_{\nu}^{٢}-a-m_{\pi}^{٢}\right)+a_{٢}^{E٣}\left(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٤}-a\Lambda_{\pi}^{٢}+٢\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$

$$\left.-m_{\pi}^{٢}\left(a+\Lambda_{\pi}^{٢}\right)\right)+a_{٣}^{E٣}\left(-٢\Lambda_{\pi}^{٢}-a-m_{\pi}^{٢}\right)+a_{٤}^{E٣}\left(\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}-\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$

$$\left.+٢\Lambda_{\pi}^{٢}\left(a+m_{\pi}^{٢}\right)+am_{\pi}^{٢}\right)+a_{٥}^{E٣}\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}-٤\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+٢a\left(\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٢}\right)\right)$$

$$-a_{٦}^{E٣}\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}-٤\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+٢m_{\pi}^{٢}\left(\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٢}\right)\right)=٠, \qquad (C٦٩)$$

$$a_{١}^{E٣}+a_{٢}^{E٣}\left(\Lambda_{\pi}^{٢}-٢\Lambda_{\nu}^{٢}+a+m_{\pi}^{٢}\right)+a_{٣}^{E٣}+a_{٤}^{E٣}\left(\Lambda_{\nu}^{٢}-٢\Lambda_{\pi}^{٢}-a-m_{\pi}^{٢}\right)$$

$$+a_٥^{E٣}\left(٢\left(\Lambda_\nu^٢-\Lambda_\pi^٢\right)-a\right)-a_٦^{E٣}\left(٢\left(\Lambda_\nu^٢-\Lambda_\pi^٢\right)-m_\pi^٢\right)=٠, \tag{C٧٠}$$

$$-a_٢^{E٣}+a_٤^{E٣}+a_٥^{E٣}-a_٦^{E٣}=٠. \tag{C٧١}$$

For the $a_i^{E٤}$ terms:

$$\begin{aligned}&a_١^{E٤}am_\pi^٢\Lambda_\nu^٤+a_٢^{E٤}am_\pi^٢\Lambda_\pi^٢\Lambda_\nu^٤+a_٣^{E٤}am_\pi^٢\Lambda_\pi^٤+a_٤^{E٤}am_\pi^٢\Lambda_\pi^٤\Lambda_\nu^٢\\&-a_٥^{E٤}a\Lambda_\pi^٤\Lambda_\nu^٤+a_٦^{E٤}m_\pi^٢\Lambda_\pi^٤\Lambda_\nu^٤=m_\Delta^{*٥}m^*s+m_\Delta^{*٤}m^{*٢}s-٣m_\Delta^{*٥}m^{*٣},\end{aligned} \tag{C٧٢}$$

$$\begin{aligned}&+a_١^{E٤}\left(-\Lambda_\nu^٤\left(a+m_\pi^٢\right)+٢am_\pi^٢\Lambda_\nu^٢\right)+a_٢^{E٤}\left(-\Lambda_\nu^٤\Lambda_\pi^٢\left(a+m_\pi^٢\right)\right.\\&\left.+٢am_\pi^٢\Lambda_\nu^٢\Lambda_\pi^٢-am_\pi^٢\Lambda_\nu^٤\right)-a_٣^{E٤}\left(\Lambda_\pi^٤\left(a+m_\pi^٢\right)+٢am_\pi^٢\Lambda_\pi^٢\right)\\&+a_٤^{E٤}\left(-\Lambda_\pi^٤\Lambda_\nu^٢\left(a+m_\pi^٢\right)-٢am_\pi^٢\Lambda_\pi^٢\Lambda_\nu^٢+am_\pi^٢\Lambda_\pi^٤\right)+a_٥^{E٤}\left(\Lambda_\pi^٤\Lambda_\nu^٤\right.\\&\left.+٢a\left(\Lambda_\pi^٢\Lambda_\nu^٤-\Lambda_\pi^٤\Lambda_\nu^٢\right)\right)-a_٦^{E٤}\left(\Lambda_\pi^٤\Lambda_\nu^٤+٢m_\pi^٢\left(\Lambda_\pi^٢\Lambda_\nu^٤-\Lambda_\pi^٤\Lambda_\nu^٢\right)\right)\\&=-m_\Delta^{*٥}m^*+m_\Delta^{*٤}\left(٢s-٣m^{*٢}\right),\end{aligned} \tag{C٧٣}$$

$$\begin{aligned}&a_١^{E٤}\left(\Lambda_\nu^٤-٢\Lambda_\nu^٢\left(a+m_\pi^٢\right)+am_\pi^٢\right)+a_٢^{E٤}\left(\Lambda_\nu^٤\Lambda_\pi^٢-٢\Lambda_\nu^٢\Lambda_\pi^٢\left(a+m_\pi^٢\right)\right.\\&\left.+\Lambda_\nu^٤\left(a+m_\pi^٢\right)+am_\pi^٢\Lambda_\pi^٢-٢am_\pi^٢\Lambda_\nu^٢\right)+a_٣^{E٤}\left(\Lambda_\pi^٤+٢\Lambda_\pi^٢\left(a+m_\pi^٢\right)+am_\pi^٢\right)\\&+a_٤^{E٤}\left(\Lambda_\pi^٤\Lambda_\nu^٢+٢\Lambda_\pi^٢\Lambda_\nu^٢\left(a+m_\pi^٢\right)-\Lambda_\pi^٤\left(a+m_\pi^٢\right)+am_\pi^٢\Lambda_\nu^٢-٢am_\pi^٢\Lambda_\pi^٢\right)\\&+a_٥^{E٤}\left(٢\Lambda_\nu^٢\Lambda_\pi^٤-٢\Lambda_\pi^٢\Lambda_\nu^٤-a\left(\Lambda_\pi^٤+\Lambda_\nu^٤\right)+٤a\Lambda_\nu^٢\Lambda_\pi^٢\right)-a_٦^{E٤}\left(٢\Lambda_\nu^٢\Lambda_\pi^٤-٢\Lambda_\pi^٢\Lambda_\nu^٤\right.\\&\left.-m_\pi^٢\left(\Lambda_\pi^٤+\Lambda_\nu^٤\right)+٤m_\pi^٢\Lambda_\nu^٢\Lambda_\pi^٢\right)=-m_\Delta^{*٤},\end{aligned} \tag{C٧٤}$$

$$a_{١}^{E٤}\left(٢\Lambda_{\nu}^{٢}-a-m_{\pi}^{٢}\right)+a_{٢}^{E٤}\left(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٤}-a\Lambda_{\pi}^{٢}+٢\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$
$$\left.-m_{\pi}^{٢}\left(a+\Lambda_{\pi}^{٢}\right)\right)+a_{٣}^{E٤}\left(-٢\Lambda_{\pi}^{٢}-a-m_{\pi}^{٢}\right)+a_{٤}^{E٤}\left(\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}-\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$
$$\left.+٢\Lambda_{\pi}^{٢}\left(a+m_{\pi}^{٢}\right)+am_{\pi}^{٢}\right)+a_{٥}^{E٤}\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}-٤\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+٢a\left(\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٢}\right)\right)$$
$$-a_{٦}^{E٤}\left(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}-٤\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+٢m_{\pi}^{٢}\left(\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٢}\right)\right)=٠, \qquad (C٧٥)$$

$$a_{١}^{E٤}+a_{٢}^{E٤}\left(\Lambda_{\pi}^{٢}-٢\Lambda_{\nu}^{٢}+a+m_{\pi}^{٢}\right)+a_{٣}^{E٤}+a_{٤}^{E٤}\left(\Lambda_{\nu}^{٢}-٢\Lambda_{\pi}^{٢}-a-m_{\pi}^{٢}\right)$$
$$+a_{٥}^{E٤}\left(٢\left(\Lambda_{\nu}^{٢}-\Lambda_{\pi}^{٢}\right)-a\right)-a_{٦}^{E٤}\left(٢\left(\Lambda_{\nu}^{٢}-\Lambda_{\pi}^{٢}\right)-m_{\pi}^{٢}\right)=٠, \qquad (C٧٦)$$

$$-a_{٢}^{E٤}+a_{٤}^{E٤}+a_{٥}^{E٤}-a_{٦}^{E٤}=٠. \qquad (C٧٧)$$

For the $a_{i}^{E٥}$ terms:

$$a_{١}^{E٥}am_{\pi}^{٢}\Lambda_{\nu}^{٤}+a_{٢}^{E٥}am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}+a_{٣}^{E٥}am_{\pi}^{٢}\Lambda_{\pi}^{٤}+a_{٤}^{E٥}am_{\pi}^{٢}\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}$$
$$-a_{٥}^{E٥}a\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}+a_{٦}^{E٥}m_{\pi}^{٢}\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}=sm^{*٦}, \qquad (C٧٨)$$

$$+a_{١}^{E٥}\left(-\Lambda_{\nu}^{٤}\left(a+m_{\pi}^{٢}\right)+٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\right)+a_{٢}^{E٥}\left(-\Lambda_{\nu}^{٤}\Lambda_{\pi}^{٢}\left(a+m_{\pi}^{٢}\right)\right.$$
$$\left.+٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}-am_{\pi}^{٢}\Lambda_{\nu}^{٤}\right)-a_{٣}^{E٥}\left(\Lambda_{\pi}^{٤}\left(a+m_{\pi}^{٢}\right)+٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\right)$$
$$+a_{٤}^{E٥}\left(-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\left(a+m_{\pi}^{٢}\right)-٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+am_{\pi}^{٢}\Lambda_{\pi}^{٤}\right)+a_{٥}^{E٥}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}\right.$$
$$\left.+٢a\left(\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\right)\right)-a_{٦}^{E٥}\left(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}+٢m_{\pi}^{٢}\left(\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\right)\right)$$
$$=٩m^{*٦}-s^{٢}m^{*٢}-٦sm^{*٤}, \qquad (C٧٩)$$

$$
\begin{aligned}
& a_١^{E٥}\left(\Lambda_\nu^٤ - ٢\Lambda_\nu^٢\left(a+m_\pi^٢\right)+am_\pi^٢\right)+a_٢^{E٥}\left(\Lambda_\nu^٤\Lambda_\pi^٢ - ٢\Lambda_\nu^٢\Lambda_\pi^٢\left(a+m_\pi^٢\right)\right. \\
& +\Lambda_\nu^٤\left(a+m_\pi^٢\right)+am_\pi^٢\Lambda_\pi^٢ - ٢am_\pi^٢\Lambda_\nu^٢\left.\right)+a_٣^{E٥}\left(\Lambda_\pi^٤ + ٢\Lambda_\pi^٢\left(a+m_\pi^٢\right)+am_\pi^٢\right) \\
& +a_٤^{E٥}\left(\Lambda_\pi^٤\Lambda_\nu^٢ + ٢\Lambda_\pi^٢\Lambda_\nu^٢\left(a+m_\pi^٢\right)-\Lambda_\pi^٤\left(a+m_\pi^٢\right)+am_\pi^٢\Lambda_\nu^٢ - ٢am_\pi^٢\Lambda_\pi^٢\right) \\
& +a_٥^{E٥}\left(٢\Lambda_\nu^٢\Lambda_\pi^٤ - ٢\Lambda_\pi^٢\Lambda_\nu^٤ - a\left(\Lambda_\pi^٤+\Lambda_\nu^٤\right)+٤a\Lambda_\nu^٢\Lambda_\pi^٢\right)-a_٦^{E٥}\left(٢\Lambda_\nu^٢\Lambda_\pi^٤ - ٢\Lambda_\pi^٢\Lambda_\nu^٤\right. \\
& -m_\pi^٢\left(\Lambda_\pi^٤+\Lambda_\nu^٤\right)+٤m_\pi^٢\Lambda_\nu^٢\Lambda_\pi^٢\left.\right)=٧sm^{*٢} - ١٢m^{*٤}\ ,
\end{aligned}
$$

(C٨٠)

$$
\begin{aligned}
& a_١^{E٥}\left(٢\Lambda_\nu^٢ - a - m_\pi^٢\right)+a_٢^{E٥}\left(٢\Lambda_\nu^٢\Lambda_\pi^٢ - \Lambda_\nu^٤ - a\Lambda_\pi^٢ + ٢\Lambda_\nu^٢\left(a+m_\pi^٢\right)\right. \\
& -m_\pi^٢\left(a+\Lambda_\pi^٢\right)\left.\right)+a_٣^{E٥}\left(-٢\Lambda_\pi^٢ - a - m_\pi^٢\right)+a_٤^{E٥}\left(\Lambda_\pi^٤ - ٢\Lambda_\pi^٢\Lambda_\nu^٢ - \Lambda_\nu^٢\left(a+m_\pi^٢\right)\right. \\
& +٢\Lambda_\pi^٢\left(a+m_\pi^٢\right)+am_\pi^٢\left.\right)+a_٥^{E٥}\left(\Lambda_\pi^٤+\Lambda_\nu^٤ - ٤\Lambda_\pi^٢\Lambda_\nu^٢ + ٢a\left(\Lambda_\pi^٢ - \Lambda_\nu^٢\right)\right) \\
& -a_٦^{E٥}\left(\Lambda_\pi^٤+\Lambda_\nu^٤ - ٤\Lambda_\pi^٢\Lambda_\nu^٢ + ٢m_\pi^٢\left(\Lambda_\pi^٢ - \Lambda_\nu^٢\right)\right)=٦m^{*٢}, \qquad (C٨١)
\end{aligned}
$$

$$
\begin{aligned}
& a_١^{E٥}+a_٢^{E٥}\left(\Lambda_\pi^٢ - ٢\Lambda_\nu^٢ + a + m_\pi^٢\right)+a_٣^{E٥}+a_٤^{E٥}\left(\Lambda_\nu^٢ - ٢\Lambda_\pi^٢ - a - m_\pi^٢\right) \\
& \quad +a_٥^{E٥}\left(٢\left(\Lambda_\nu^٢ - \Lambda_\pi^٢\right)-a\right)-a_٦^{E٥}\left(٢\left(\Lambda_\nu^٢ - \Lambda_\pi^٢\right)-m_\pi^٢\right)=٠, \qquad (C٨٢)
\end{aligned}
$$

$$
-a_٢^{E٥}+a_٤^{E٥}+a_٥^{E٥}-a_٦^{E٥}=٠. \qquad (C٨٣)
$$

For the $a_i^{E٦}$ terms:

$$
\begin{aligned}
& a_١^{E٦}am_\pi^٢\Lambda_\nu^٤ + a_٢^{E٦}am_\pi^٢\Lambda_\pi^٢\Lambda_\nu^٤ + a_٣^{E٦}am_\pi^٢\Lambda_\pi^٤ + a_٤^{E٦}am_\pi^٢\Lambda_\pi^٤\Lambda_\nu^٢ \\
& \qquad -a_٥^{E٦}a\Lambda_\pi^٤\Lambda_\nu^٤ + a_٦^{E٦}m_\pi^٢\Lambda_\pi^٤\Lambda_\nu^٤ = -m_\Delta^{*٦}m^{*٢} - ٢m^{*٨}, \qquad (C٨٤)
\end{aligned}
$$

$$
+a_١^{E٦}\left(-\Lambda_\nu^٤\left(a+m_\pi^٢\right)+٢am_\pi^٢\Lambda_\nu^٢\right)+a_٢^{E٦}\left(-\Lambda_\nu^٤\Lambda_\pi^٢\left(a+m_\pi^٢\right)\right.
$$

$$+\,٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}-am_{\pi}^{٢}\Lambda_{\nu}^{٤}\Big)-a_{٣}^{E٦}\Big(\Lambda_{\pi}^{٤}\big(a+m_{\pi}^{٢}\big)+٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Big)$$

$$+\,a_{٤}^{E٦}\Big(-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\big(a+m_{\pi}^{٢}\big)-٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+am_{\pi}^{٢}\Lambda_{\pi}^{٤}\Big)+a_{٥}^{E٦}\Big(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}$$

$$+\,٢a\big(\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\big)\Big)-a_{٦}^{E٦}\Big(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٤}+٢m_{\pi}^{٢}\big(\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}\big)\Big)$$

$$=٠, \qquad \text{(C٨٥)}$$

$$a_{١}^{E٦}\Big(\Lambda_{\nu}^{٤}-٢\Lambda_{\nu}^{٢}\big(a+m_{\pi}^{٢}\big)+am_{\pi}^{٢}\Big)+a_{٢}^{E٦}\Big(\Lambda_{\nu}^{٤}\Lambda_{\pi}^{٢}-٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\big(a+m_{\pi}^{٢}\big)$$

$$+\,\Lambda_{\nu}^{٤}\big(a+m_{\pi}^{٢}\big)+am_{\pi}^{٢}\Lambda_{\pi}^{٢}-٢am_{\pi}^{٢}\Lambda_{\nu}^{٢}\Big)+a_{٣}^{E٦}\Big(\Lambda_{\pi}^{٤}+٢\Lambda_{\pi}^{٢}\big(a+m_{\pi}^{٢}\big)+am_{\pi}^{٢}\Big)$$

$$+\,a_{٤}^{E٦}\Big(\Lambda_{\pi}^{٤}\Lambda_{\nu}^{٢}+٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}\big(a+m_{\pi}^{٢}\big)-\Lambda_{\pi}^{٤}\big(a+m_{\pi}^{٢}\big)+am_{\pi}^{٢}\Lambda_{\nu}^{٢}-٢am_{\pi}^{٢}\Lambda_{\pi}^{٢}\Big)$$

$$+\,a_{٥}^{E٦}\Big(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}-a\big(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}\big)+٤a\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\Big)-a_{٦}^{E٦}\Big(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٤}$$

$$-\,m_{\pi}^{٢}\big(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}\big)+٤m_{\pi}^{٢}\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}\Big)=-s^{٢}, \qquad \text{(C٨٦)}$$

$$a_{١}^{E٦}\Big(٢\Lambda_{\nu}^{٢}-a-m_{\pi}^{٢}\Big)+a_{٢}^{E٦}\Big(٢\Lambda_{\nu}^{٢}\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٤}-a\Lambda_{\pi}^{٢}+٢\Lambda_{\nu}^{٢}\big(a+m_{\pi}^{٢}\big)$$

$$-\,m_{\pi}^{٢}\big(a+\Lambda_{\pi}^{٢}\big)\Big)+a_{٣}^{E٦}\Big(-٢\Lambda_{\pi}^{٢}-a-m_{\pi}^{٢}\Big)+a_{٤}^{E٦}\Big(\Lambda_{\pi}^{٤}-٢\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}-\Lambda_{\nu}^{٢}\big(a+m_{\pi}^{٢}\big)$$

$$+\,٢\Lambda_{\pi}^{٢}\big(a+m_{\pi}^{٢}\big)+am_{\pi}^{٢}\Big)+a_{٥}^{E٦}\Big(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}-٤\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+٢a\big(\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٢}\big)\Big)$$

$$-\,a_{٦}^{E٦}\Big(\Lambda_{\pi}^{٤}+\Lambda_{\nu}^{٤}-٤\Lambda_{\pi}^{٢}\Lambda_{\nu}^{٢}+٢m_{\pi}^{٢}\big(\Lambda_{\pi}^{٢}-\Lambda_{\nu}^{٢}\big)\Big)=-٢s,$$

(C٨٧)

$$a_{١}^{E٦}+a_{٢}^{E٦}\Big(\Lambda_{\pi}^{٢}-٢\Lambda_{\nu}^{٢}+a+m_{\pi}^{٢}\Big)+a_{٣}^{E٦}+a_{٤}^{E٦}\Big(\Lambda_{\nu}^{٢}-٢\Lambda_{\pi}^{٢}-a-m_{\pi}^{٢}\Big)$$

$$+\,a_{٥}^{E٦}\Big(٢\big(\Lambda_{\nu}^{٢}-\Lambda_{\pi}^{٢}\big)-a\Big)-a_{٦}^{E٦}\Big(٢\big(\Lambda_{\nu}^{٢}-\Lambda_{\pi}^{٢}\big)-m_{\pi}^{٢}\Big)=-١, \qquad \text{(C٨٨)}$$

$$-a_{٢}^{E٦}+a_{٤}^{E٦}+a_{٥}^{E٦}-a_{٦}^{E٦}=٠. \qquad \text{(C٨٩)}$$

C.III ***<u>Gaussian Elimination</u>***

Gaussian elimination is the first method usually presented in algebra for the solution of simultaneous linear algebraic equations in which the unknowns are eliminated by combining the equations.

A set of linear algebraic equations look like this:

$$E_١^{٠} = a_{١١}x_١ + a_{١٢}x_٢ + ... + a_{١n}x_n = b_١$$
$$E_٢^{٠} = a_{٢١}x_١ + a_{٢٢}x_٢ + ... + a_{٢n}x_n = b_٢$$
$$E_٣^{٠} = a_{٣١}x_١ + a_{٣٢}x_٢ + ... + a_{٣n}x_n = b_٣$$
.
.
.
$$E_n^{٠} = a_{n١}x_١ + a_{n٢}x_٢ + ... + a_{nn}x_n = b_n . \qquad (C٩٠)$$

Here the n unknowns x_i, i=١،٢,…,n are related by n equations. The coefficients a_{ij} with i, j=١،٢,…,n are known numbers, as are the right-hand side quantities b_i, i=١،٢,…,n.

Divide the first equation in (C٩٠) by the coefficient of $x_١$, we get

$$E_١^{١} = \frac{E_١^{٠}}{a_{١١}} = x_١ + \frac{a_{١٢}}{a_{١١}}x_٢ + ... + \frac{a_{١n}}{a_{١١}}x_n = \frac{b_١}{a_{١١}} \cdot \qquad (C٩١)$$

Then multiply Eq. (C٩١) by the coefficient of $x_١$ in $E_٢^{٠}$ and subtract the resulting equation from $E_٢^{٠}$ i.e.,

$$E_٢^{١} = E_٢^{٠} - a_{٢١}E_١^{١}$$
$$= \left(a_{٢٢} - a_{٢١}\frac{a_{١٢}}{a_{١١}} \right)x_٢ + \left(a_{٢٣} - a_{٢١}\frac{a_{١٣}}{a_{١١}} \right)x_٣ + ... + \left(a_{٢n} - a_{٢١}\frac{a_{١n}}{a_{١١}} \right)x_n$$
$$= b_٢ - a_{٢١}\frac{b_١}{a_{١١}}, \qquad (C٩٢)$$

and so on

$$\begin{aligned} E_n^{١} &= E_n^{٠} - a_{n١}E_١^{١} \\ &= a_{n٢}^{١}x_٢ + a_{n٣}^{١}x_٣ + \ldots + a_{nn}^{١}x_n = b_n^{١}. \end{aligned} \qquad (C٩٣)$$

After the first elimination we get new equations which may be written as:

$$\begin{aligned} E_١^{١} &= x_١ + a_{١٢}^{١}x_٢ + \ldots + a_{١n}^{١}x_n = b_١^{١} \\ E_٢^{١} &= ٠ + a_{٢٢}^{١}x_٢ + \ldots + a_{٢n}^{١}x_n = b_٢^{١} \\ E_٣^{١} &= ٠ + a_{٣٢}^{١}x_٢ + \ldots + a_{٣n}^{١}x_n = b_٣^{١} \\ &. \\ &. \\ &. \\ E_n^{١} &= ٠ + a_{n٢}^{١}x_٢ + \ldots + a_{nn}^{١}x_n = b_n^{١}. \end{aligned} \qquad (C٩٤)$$

Considering the second equation in (C٩٤) and repeating the same steps to eliminate $x_٢$ from all the equations following this second equation.

Repeating the same procedure $(n - ١)$ times yields

$$\begin{aligned} & x_١ + a_{١٢}^{١}x_٢ + a_{١٣}^{١}x_٣ + \ldots + a_{١n}^{١}x_n = b_١^{١} \\ & ٠ + x_٢ + a_{٢٣}^{٢}x_٣ + \ldots + a_{٢n}^{٢}x_n = b_٢^{٢} \\ & ٠ + ٠ + x_٣ + a_{٣٤}^{٣}x_٤ + \ldots + a_{٣n}^{٣}x_n = b_٣^{٣} \\ & . \\ & . \\ & . \\ & ٠ + ٠ + ٠ + \ldots + \ldots + \ldots + a_{nn}^{n-١}x_n = b_n^{n-١}. \end{aligned} \qquad (C٩٥)$$

After the triangular set of equations has been obtained, the last equation in this equivalent set yields the value x_n directly as

$$x_n = \frac{b_n^{n-١}}{a_{nn}^{n-١}}. \qquad (C٩٦)$$

This value is then substituted into the next-to-the last equation of the triangular set to obtain a value of $x_{n-١}$ as

$$x_i = \frac{b_i - \sum_{j=i+١}^{n} a_{ij} x_j}{a_{ii}}, \qquad (C٩٧)$$

$i = n-١, n-٢, \ldots, ١.$

<u>The subprogram elimination is given by:</u>

```
      subroutine Gauss(A,B,NDIM,Neq)
c     program to solve Ax=B by Gaussian elimination
      implicit none
      integer NDIM,Neq
      real*٨ A(NDIM, NDIM), B(NDIM)
      real*٨ D
      integer N١,Np,N,J,I
      N١= Neq-١
c     Begin Elimination
      Do ٥٠٠ N=١,N١
      if (A(N,N).eq.٠,٠) goto ٩٩٩
      D=١,٠/A(N,N)
      Np=N+١
```

```
            Do ١٠ J=Np, Neq
١٠          A(N,J)=D* A(N,J)
            B(N)=D* B(N)
            Do ٥٠٠ I= Np, Neq
            Do ٤٠٠ J= Np, Neq
٤٠٠         A(I,J)= A(I,J)- A(I,N)* A(N,J)
٥٠٠         B(I)= B(I)- A(I,N)*B(N)
C           Begin back substitution
            B(Neq)= B(Neq)/A(Neq, Neq)
            Do ٧٠٠ I=١,N١
            N= Neq-I
            Np=N+١
            Do ٦٠٠ J=Np,Neq
٦٠٠         B(N)= B(N)- A(N,J)* B(J)                !output
٧٠٠         Continue
            return
٩٩٩         Write (٦،١٠٠٠) N
            N=١٠
            A(١٠،١٠)=٥٠٠,٠
            return
١٠٠٠        format ('***zero diagonal element arises in equation',I٤)
            stop
            end
```

C.IV Bisection Method

Bisection is the method that can be used to find the root of equation numerically.

Over some interval the function is known to pass through zero because it changes sign. Evaluate the function at the interval's midpoint and examine its sign. Use the midpoint to replace whichever limit has the same sign. After each iteration the bounds containing the root decrease by a factor of two. If after *n* iterations the root is known to be within an interval of size ε_n, then after the next iteration it will be bracketed within an interval of size

$$\varepsilon_{n+١} = \varepsilon_n / ٢ \qquad (C٩٨)$$

The following function subprogram generates the scattering angle $\cos\theta$ through bisection method.

```
          function costhcoll (s,m)
C         input:s, m: characteristics of the ingoing channel.
C         output: cos (theta)
          x=rndm(-١)
          dct=٢,٠
          costhcoll=-١,٠
          do j=١،١٢
          dct=٠,٥*dct
          ct= costhcoll+dct
          if (func (s,m,ct).le.x) costhcoll=ct                ! Eq.(C١)
          end do
          return
          end
```

[١] Andersson B., Gustafson G. and Pi H., Z. Phys. C ٥٧, ٤٨٥

(١٩٩٣).

List of references

[١] Andersson B., Gustafson G. and Pi H., Z. Phys. C ٥٧, ٤٨٥

(١٩٩٣).

[٢] Ben-Hao S. and An T., Phys. Rev. C **٥٥**, ٢٠١٠ (١٩٩٧).

[٣] Werner K., Phys. Rep. ٢٣٢, ٨٧ (١٩٩٣).

[٤] Wang X. N., Phys. Rep. **٢٨٠**, ٢٨٧ (١٩٩٧).

[٥] Capella A., Sukhatme U., C. I. Tan and Tran Thanh Van J., Phys. Rep. **٢٣٦**, ٢٢٥ (١٩٩٤).

[٦] Shor A. and Longacre R., Phys. Lett. B **٢١٨**, ١٠٠ (١٩٨٩).

[٧] Jeon S. and Kapusta J., Phys. Rev. C **٥٦**, ٤٦٨ (١٩٩٧).

[٨] Geiger K., Phys. Rep. **٢٥٨**, ٢٣٨ (١٩٩٥).

[٩] Zhang B., Gyulassy M. and Pang Y., Phys. Rev. C **٥٨**, ١١٧٥ (١٩٩٨).

[١٠] Geiger K., Nucl. Phys. **A٦٣٨**, ٥٥١c (١٩٩٨).

[١١] Sorge H., Mattiello R., Jahns A., Stocker H. and Greiner W., Phys. Lett. B **٢٧١**, ٣٧ (١٩٩١).

[١٢] Sorge H., Phys. Rev. C **٥٢**, ٣٢٩١ (١٩٩٥).

[١٣] Bravina L., Csernai L. P., Levai P., and Dtrottman D., Phys. Rev. C **٥١**, ٢١٦١ (١٩٩٤).

[١٤] Pang Y., Schlagel T. J., Kahana S. H., Nucl. Phys. **A٥٤٤**, ٤٣٥c (١٩٩٢).

[١٥] Li B. A. and Ko C. M., Phys. Rev. C **٥٨**, R١٣٨٢ (١٩٩٨).

[١٦] Winckelmann L. A. *et al*., Nucl. Phys. **A٦١٠**, ١١٦c (١٩٩٦).

[١٧] Bass S. A. *et al.*, Prog. Part. Nucl. Phys. **٤١**, ٢٢٥ (١٩٩٨).

[١٨] Ehehalt W. and Cassing W., Nucl. Phys. **A٦٠٢**, ٤٤٩ (١٩٩٦).

[١٩] Aichelin J., Phys. Rep. **٢٠٢**, ٢٣٣ (١٩٩١), and references therein.

[٢٠] Leray S. *et al*, Phys. Rev. C **٦٥**, ٠٤٤٦٢١ (٢٠٠٢).

[٢١] Cugnon J., Volant C., and Vuiller S., Nucl. Phys. **A٦٢٠**, ٥٧٥ (١٩٩٧).

[٢٢] Boudard A., Cugnon J., Leray S., and Volant C., Phys. Rev. C ٦٦, ٠٤٤٦١٥ (٢٠٠٢).

[٢٣] Engel A., Tanaka E. I., Maruyama T., Ono A., and Hoiuchi H., Phys. Rev. C **٥٢**, ٣٢٣١ (١٩٩٥).

[٢٤] Prout D. L. *et al*., Phys. Rev. C **٥٢**, ٢٢٨ (١٩٩٥).

[٢٥] Amian W. B. *et al*., Phys. Rev. C **٤٧**, ١٦٤٧ (١٩٩٣).

[٢٦] Abdel-Waged Kh., Abdel-Hafiz A., and Uzhinskii V. V., J. Phys. G **٢٦**, ١١٠٥ (٢٠٠٠).

[٢٧] Abdel-Waged Kh., Phys. Rev. C ٦٧, ٠٢٤٩٠١ (٢٠٠٣).

[٢٨] Bertsch G. F. and Das Gupta S., Phys. Rep. **١٦٠**, ١٨٩ (١٩٨٨), and references therein.

[٢٩] Sahu P. K. and Cassing W., Nucl. Phys. **A٧١٢**, ٣٥٧ (٢٠٠٢).

[٣٠] Gudima K. K. and Toneev V. D., Yad. Fiz. **٢٧**, ٦٧ (١٩٧٨).

[٣١] Cugnon J., Phys. Rev. C **٢٢**, ١٨٨٥ (١٩٨٠).

[٣٢] Cugnon J., Mizutani T., and Vandermeulen J., Nucl. Phys. **A٣٥٢**, ٥٠٥ (١٩٨١).

[٣٣] Particle Data Group, Barnnett R. M. et al., Phys. Rev. D **٥٤**, ١ (١٩٩٦).

[٣٤] Andersson B. et al., Nucl. Phys. **B٢٨١**, ٢٨٩ (١٩٨٧).

[٣٥] Cugnon J., Leray S., Mrtinez E., Patin Y., and Vuiller S., Phys. Rev. C **٥٦**, ٢٤٣١ (١٩٩٧).

[٣٦] Mao G., Li Z., Zhuo Y., Han Y., Yu Z., and Sano M., Z. Phys. A **٣٤٧**, ١٧٣ (١٩٩٤).

[٣٧] Mao G., Li Z., Zhuo Y., and Han Y., Phys. Rev. C **٤٩**, ٣١٣٧ (١٩٩٤).

[٣٨] Hama S., Clark B.C., Cooper E.D., Sherif H.S., Mercer R.L.: Phys. Rev. C **٤١**, ٢٧٣٧ (١٩٩٠).

[٣٩] Renberg P.U., Measday D.F., Pepin M., Schwaller P., Favier B., Richard-Serre C.: Nucl. Phys. **A١٨٣**, ٨١ (١٩٧٢).

[٤٠] Mao G., Li Z., and Zhuo Y., Phys. Rev. C **٥٣**, ٢٩٣٣ (١٩٩٦).

[٤١] Abdel-Waged Kh., Phys. Rev. C ٦٧, ٠٦٤٦١٠ (٢٠٠٣).

[٤٢] Hartnack C., Puri R. K., Aichelin J., Konopka J., Bass S. A., Stocker H., and Greiner W., Eur. Phys. J. A ١, ١٥١ (١٩٩٨).

[٤٣] Weisskopf V., Phys. Rev. ٥٢, ٢٩٥ (١٩٣٧).

[٤٤] Bugg D. V. et al., Phys. Rev. ١٣٣, B١٠١٧ (١٩٦٤).

[٤٥] Parel R. E. and Liechtenstein H., Report No. LA-UR-٨٩-٣٠١٤, Los Alamos National Laboratory (١٩٨٩).

[٤٦] William H. p., Saul A. T., William T. V., and Brian P. F., Numerical Recipes in Fortran, second edition, Cambridge university press ١٩٨٦, ١٩٩٢.

الشكــر والعرفــان

أ ـد ا تعا أن من علي بإ ام هذا العمل بتوفيقه وعظيم كرمه، فمثل هذا العمل ينجز بشكل فردي لذلك أود أن أتقدم بصادق شكري وتقديري للمشرف على الرسالة سعادة الدكتور خالد عبدالواجد على توجيهاته وإرشاده ونقده البناء والذي أ ر عن هذا العمل.

كما أتقدم بشكري للمشترك UrQMD و اصة الدكتور في لتزويدنا ب نامج الـ UrQMD وال وفيسور لاراي لتزويدنا بالبيانات العملية.

وبكل التقدير أشكر رئيسة قسم العلوم والرياضيات بكلية ا علمات و القائم عليها و رئيس قسم الفيزياء امعة أم القرى على دعمهم و حسن تعاو م.

وكل الشكر لوالدي العزيزين وأخي وأخوا على حبهم ودعمهم .

لكم جميعا أهدي بحثي

ملخـص البحـث

خصص هذا البحث لدراسة تأث الوسط على تصادم بروتون مع نواة عند الطاقات ا توسـطة (١-٦ جيجا الكترون فولت) وذلك ضمن وذج حركة ا زيء الكمي النسبوي (UrQMD).

إن تصادم بروتون مع نواة عند الطاقات ا توسطة يؤدي إ تكون قمتان طيف النيوترونات عند الزوايا الأمامية تعرفان بالقمم شبه ا رنة وشبه غ ا رنة، حيث تقع القمة شبه ا رنة عند طاقة قريبة من طاقة السقوط وهي نا ة عن الاستطارة الأحادية ا رنة لتصادم بروتون- نيوترون. أما القمة شـبه الغ مرنة فهي تقع قبل طاقة السقوط وا ٣٠٠ ميجا الكترون فولت وتكون مصاحبة للحالة ا ثارة لتكون جسيم Δ الرني تصادم بروتون- نيوترون غ ا رن.

نظرا لأنه يتم الوصول للوصف النظري الشامل لكل من موقع وشدة هات القمت , وجدنا أنه من ا ناسب العمل على إدخال ما يسمى بتأث الوسط ضمن مكونات وذج حركة ا زيء الكمي النسبوي. هذا التأث يتضمن تأث ا هد ب النيوكلونات وتأث التوزيع الزاوي ا عتمد على الوسـط لكل من التفاعلات التالية: $NN - NN$ و $N\Delta - NN$ و $NN - N\Delta$. وقـد ا ـصول على سن كب وصف موقع وشدة القمت نتيجة أخذ هذه التأث ات الاعتبار.

المملكــة العربيــة السعوديـــــة
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كليــة العلوم التطبيقيـة
قسم الفيزيـاء



دراسة التفاعلات النووية المستحثة بالنيوكلونات عند الطاقات المتوسطة

أ. تأثيرات الوسط النووي

رسالة مقدمة لقسم الفيزياء بكلية العلوم التطبيقية بجامعة أم القرى كمتطلب تكميلي لبرنامج درجة الماجستير في الفيزياء

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الأستاذ المشارك في الفيزياء

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